OPTIMAL CONTROL
FOR THE ROLLING PULLOUT MANEUVER
OF A MODERN FIGHTER AIRCRAFT

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CERTIFICATE

Certified that this work on "Optimal Control for the Rolling Pullout Maneuver of a Modern Fighter Aircraft" has been carried out under my supervision and that this has not been submitted elsewhere for a degree.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

α	Incremental angle of attack; angle between X stability axis and the projection of the relative wind on the plane of symmetry. rad.
β	Sideslip angle; angle between relative wind and plane of symmetry. rad.
p,q,r	Incremental angular velocities about X,Y and Z stability axes respectively. rad/sec.
ϕ,θ,ψ	Incremental angular displacements about X,Y and Z stability axes respectively. rad.
u,v,w	Incremental linear velocities about X,Y and Z stability axes respectively. ft/sec.
W_t	Weight of the airplane in lbs.
S	Wing area. ft^2.
I_x,I_y,I_z	Moments of Inertia about the X,Y and Z stability axes. slug ft^2.
δ_e	Elevator deflection angle. degrees or rad.
δ_a	Sum of left aileron deflection and right aileron deflection. degrees or rad.
Z_α	Rate of change of angle of attack due to unit change in α.
Y_β	Rate of change of β due to unit change in β.
M_α	Pitch acceleration due to unit change in α.
M_q	Pitch acceleration due to unit pitch rate.
$M_{\delta e}$	Pitch acceleration due to unit elevator deflection.
L_p	Roll acceleration due to unit roll rate.
$L_\beta(\alpha)$	Roll acceleration due to unit β.
$L_{\delta a}(\alpha)$	Roll acceleration due to unit aileron deflection.

N_{β}	Yaw acceleration due to unit β.
N_r	Yaw acceleration due to unit yaw rate.
$N_p(\alpha)$	Yaw acceleration due to unit roll rate.
$N_{\delta a}(\alpha)$	Yaw acceleration due to unit aileron deflection.

Note: $L_{\beta}(\alpha), L_{\delta a}(\alpha), N_p(\alpha), N_{\delta a}(\alpha)$ are nonlinear functions of α.

Subscript "max" for any variable represents its maximum value.

Dot over any variable represents its time derivative.

ABSTRACT

Rolling pullout is one of the important tactical maneuvers which can be used in offensive as well as defensive roles. The problem of designing an optimal controller for performing this maneuver with a modern fighter aircraft is formulated. The effect of inertial cross coupling terms and the need for their inclusion in a mathematical model of a modern aircraft are clearly brought out. A five degree of freedom nonlinear model including these coupling terms and other nonlinear aerodynamic terms is proposed for the maneuver problem. The maneuver problem is transformed to an optimal control problem with the formulation of a suitable performance index and state model. The Gradient method of solving the optimal control problem and some computational aspects peculiar to this problem are discussed. The problem is solved for three different flight conditions for a modern swept wing fighter aircraft and the resulting optimal controls and optimal trajectories are analysed.

CHAPTER I

INTRODUCTION

The control problems associated with modern aircraft are quite diversified in nature. This depends in general on the type of the aircraft viz. fighter, bomber, transport etc. and on the speed, whether subsonic, transonic or supersonic.

In early days, autopilots were designed mainly for transport aircraft with a view to maintain the attitude and heading of the aircraft. But with the advent of high performance jet aircraft, the tactical requirements of high maneuverability invariably ended up in a design which had many control problems. This is accentuated by the fact that stability and maneuverability are two conflicting requirements in airframe design; one has to be sacrificed for the benefit of the other. The tactical requirements give preference to maneuverability over stability at the airframe design stage. But stability of an aircraft is equally important and this is taken care of by suitable automatic control systems. Thus automatic control systems have become an integral feature of the aircraft.

Basically the aircraft control systems are of two kinds : (1) To assist the pilot in handling the aircraft efficiently without strain, by stabilizing any unstable

motions. (2) To perform a specified task or maneuver. The conventional autopilots, pitch dampers and yaw dampers belong to the former class; and Instrument landing systems and Maneuver demand systems belong to the latter. The maneuver demand systems are those which perform a desired maneuver in a specified way. In the case of transport aircraft, this may be an automatic landing system. In a fighter, this system may be designed for performing a complicated maneuver which the pilot may not be able to perform as efficiently as a control system, which can have decidedly better accuracy, ~~judgement~~ and speed of response.

Traditionally, the aircraft control systems have been designed using classical transfer function methods, where the overall requirements such as rise time, bandwidth etc. can be satisfied in ever so many ways and it was left to the discretion of the designer to choose the system he feels best. On the other hand, application of Modern control techniques has distinct advantages in the design of a control system for performing a tactical maneuver. This is due to the fact that the requirements and various physical constraints of the maneuver can be very precisely specified and the application of optimal control theory results in a performance which is optimum in the sense that the performance cannot be improved upon any further without altering the problem itself.

1.2 SCOPE OF THE THESIS

The thesis problem is concerned with designing an optimal control system for performing a very important tactical maneuver, namely the rolling pullout maneuver.

The rolling pullout is one of those maneuvers which can be used in both offensive and defensive roles. In defensive role, it may be required for evasion and escape from an attacking fighter aircraft at close range. In offensive role, this maneuver appears to be necessary during attacks with high closure rates, in order to make aiming corrections and to achieve satisfactory breakaway after release of the armament. The maneuver is usually initiated from a straight and level flight. At the end of the maneuver, the aircraft has rolled through about 80° and is subjected to high normal acceleration. This puts the aircraft in a tight turn which is the object of this maneuver. The maneuver is normally performed by the pilot by first rolling the aircraft quickly to one side through the desired angle by a step movement of the aileron; and then applying a nose up elevator which puts the aircraft in a tight turn with high normal acceleration. Obviously, a maneuver like this, involving the operation of two control surfaces, can be performed in innumerable ways; the optimal way being the one in which the maneuver is performed in the shortest time with maximum normal

The object of this thesis is to find an optimal control law which will roll a given aircraft through a desired angle in minimum time and simultaneously achieve maximum normal acceleration at the end of this time. This maneuver problem is formulated in this thesis as a bicriterion optimal control problem subject to (1) constraints on the aircraft parameters like angle of attack, side slip angle, roll rate etc., (2) constraints on available control power and (3) a specified attitude for the aircraft at the end of the maneuver.

The basic mathematical model used for the aircraft for this problem is that due to Rhoads [1] with a few simplifications. Since the model chosen is nonlinear, the resulting optimal control problem cannot be solved analytically. Hence, a numerical method of solving the problem is indicated. Using this method, an example problem is solved for a swept wing fighter aircraft whose stability derivatives are known for three flight conditions. Implementation of the resulting optimal control law and extension of this method to take care of other flight conditions are discussed. Scope for further research in this area is also indicated towards the end.

1.3 BRIEF OUTLINE OF THE CHAPTERS

A brief description of the contents of the various chapters is presented below.

Chapter II discusses the effect of inertial cross coupling in modern aircraft and emphasizes the need for including the terms due to inertial coupling, in the equations of motion of the aircraft. The effect of neglecting the coupling terms on the aircraft response is brought out by plotting the aircraft response during a roll maneuver with and without these terms.

The physical requirements of the rolling pullout maneuver and the associated control problem are stated in Chapter III. The chapter presents a workable model for the aircraft to suit the maneuver. The various assumptions made and the justifications in arriving at such a workable model are discussed.

In Chapter IV, a suitable performance criterion is formulated and the aircraft maneuver problem is transformed into an optimal control problem with the help of a proper state model. The gradient method of solving the optimal control problem is indicated.

Some computational aspects peculiar to this problem are brought out in Chapter V. The computer flow chart and functions of various subroutines are presented. The maneuver problem is solved for a modern fighter aircraft for three different flight conditions and the results are plotted. A few conclusions based on

these results are given. The implementation aspects of the optimal control and extension of the method to other flight conditions are also discussed.

The thesis concludes with an indication of the topics for further research.

CHAPTER II

INERTIAL COUPLING AND ASSOCIATED CONTROL PROBLEM IN A MODERN FIGHTER AIRCRAFT

2.1 INTRODUCTION

With the increase in speed of the aircraft, especially after World War II, the structural and control problems associated with the aircraft have correspondingly become more complex. One such problem is that due to the inertial coupling between lateral and longitudinal modes in a high speed fighter aircraft. Due to this phenomenon, application of rolling moment to a modern aircraft results in some yawing and pitching moments as well which, in many cases may cause divergence in pitch or yaw depending on the magnitude of rolling moment.

In this chapter, a brief review of the research done on the dynamics of an inertially coupled aircraft is given. The effect of neglecting the coupling terms in a maneuver involving rolling, are clearly brought out by comparing the plots of response of a modern fighter aircraft, with and without the coupling terms taken into account.

2.2 INERTIAL CROSS COUPLING

With increase in speed, the aircraft configuration changed considerably from its prewar shape, to one with short

slender wings with high sweepback and with most of its mass concentrated in the fuselage. The shift of weight from wings to the fuselage resulted in a decrease in moment of inertia of the aircraft about the longitudinal axis or the so called X axis, and the inertia about the lateral and directional axes increased. This phenomenon increased the coupling between lateral and long itudinal equations of motion and can be seen by examining the basic moment equations of an aircraft which are,

$$L = \dot{p}I_x - \dot{r}I_{xz} + qr(I_z - I_y) - pqI_{xz}$$

$$M = \dot{q}I_y - pr(I_z - I_x) + (p^2 - r^2)\, I_{xz}$$

$$N = \dot{r}I_z - \dot{p}I_{xz} + pq(I_y - I_x) + qrI_{xz} \qquad (2.2.1)$$

where L, M, N are rolling, pitching and yawing moments respectively; p, q, r and I_x, I_y, I_z are angular velocities and moments of inertia respectively about OX, OY and OZ axes. I_{xz} is product of inertia term.

As I_x becomes much smaller than I_y and I_z, the moment of inertia difference terms $(I_z - I_x)$ and $(I_y - I_x)$ become large. Application of rolling moment to the aircraft results in some yawing moment and the term $pr(I_z-I_x)$ may become large enough to cause an uncontrollable pitching moment. Hence this phenomenon is also known as roll coupling. Moreover, when an aircraft rolls about an axis

other than velocity vector, for example about the longitudinal axis with some angle of attack, the angle of attack after 90° roll is a side slip angle; after 180° of roll it is a negative angle of attack and after 270° it is again a side slip angle. This interchange of angle of attack and side slip angles causes the aerodynamic moments to alternate between pitching and yawing moments which aggravates the inertial cross coupling.

The subject has received considerable attention in literature [2-11] Philips [2] considers a two degree of freedom aircraft model including inertial coupling terms. His work is mainly concerned with the stability of a rolling aircraft. He derives expressions for frequencies of oscillations of rolling aircraft (w_1 and w_2) in terms of the steady roll frequency P_o and the natural frequencies of oscillations in pitch and yaw of non-rolling aircraft ($w_{n\theta}$ and $w_{n\psi}$). The following are the results obtained by him for a rolling aircraft with low I_x.

(1) When both $w_{n\theta}$ and $w_{n\psi}$ are greater than P_o, the motion is stable. The frequencies of oscillations of rolling aircraft under this condition are,

$$\left.\begin{aligned} w_1 &= \max\{w_{n\theta}, w_{n\psi}\} \\ w_2 &= \min\{w_{n\theta}, w_{n\psi}\} \end{aligned}\right\} \quad \text{if } w_{n\theta} \neq w_{n\psi}$$

(2) When either $w_{n\theta}$ or $w_{n\psi}$ equals P_o and the other greater than P_o, the rolling aircraft is neutrally stable in one mode.

(3) When either of $w_{n\theta}$ or $w_{n\psi}$ is less than P_o and the other is greater than P_o, the rolling aircraft becomes statically unstable in one mode and performs a straight divergence.

(4) When both $w_{n\theta}$ and $w_{n\psi}$ are greater than zero but less than P_o, the rolling aircraft is again stable.

(5) When both $w_{n\theta}$ and $w_{n\psi}$ are much less than P_o, which means static longitudinal and directional stabilities approach zero, the two frequencies of rolling aircraft w_1 and w_2 both approach P_o. Physically this means that the rolling aircraft can have its axis tilted from flight path and due to its lack of static stability, will continue to roll about this tilted axis.

From the above results, the following conclusions can be drawn.

(1) Instability may be caused by very high roll rates in modern aircraft with low I_x. This instability lasts only as long as the aircraft rolls and would not therefore cause difficulty in normal flying.

(2) The roll rate for a given aileron deflection remains approximately constant for a given True air speed. But $w_{n\theta}$ and $w_{n\psi}$ decrease as the Indicated air speed is reduced with increase in altitude. Hence rolling frequency may exceed one or other oscillation frequencies at high altitudes.

(3) The above mentioned instability would not be present if pitching and yawing frequencies were equal. But this is an unrealizable solution in any aircraft since the longitudinal stability inevitably varies with change in C.G. position.

(4) The rate of divergence does not cause serious attitude change upto 90° angle of bank, but may cause serious changes in a complete 360° roll.

Rhoads and O'hara [1] have carried out exhaustive research on the effects of inertial cross coupling on tail loads during a roll pullout maneuver. The model derived by them is a six degree of freedom model which can be applied in general to any large disturbance maneuver. They analyse the roll pullout maneuver thoroughly for the F80 A* aircraft. Also in their paper on Airplane dynamics of large disturbance maneuvers, Rhoads and Schuler have presented a simplified model with five degrees of freedom to the dynamics of large disturbance maneuvers with particular emphasis on the non-linearities associated with high roll rates. [10]

The equations as given by them are as follows:

---

* F80 A is a conventional subsonic fighter aircraft with a wing loading of 40 lb/sq.ft., aspect ratio 6.4 and a leading edge sweep back of 9°.

$$
\begin{aligned}
\dot{\alpha} &= Z_{\alpha}\alpha - p\beta + q \\
\dot{\beta} &= p\alpha + Y_{\beta}\beta - r \\
\dot{p} &= L_{\beta}(\alpha)\beta + L_{p}p + qr\,(I_y - I_z)/I_x + L_{\delta a}(\alpha)\delta a \qquad (2.2.2) \\
\dot{q} &= M_{\alpha}\alpha + M_{q}q + pr\,(I_z - I_x)/I_y + M_{\delta e}\delta e \\
\dot{r} &= N_{\beta}\beta + N_{r}r + N_{p}(\alpha)p + N_{\delta a} + N_{\delta a}(\alpha)\delta a + pq(I_x - I_y)/I_z
\end{aligned}
$$

where the inertial cross coupling terms and all major non-linear terms are included. The assumptions made in arriving at this model and the justifications in using this model for the rolling pullout maneuver are discussed in detail in Chapter III.

Blakelock, while considering the effects of inertial cross coupling omits the roll transients [11]. He drops the rolling moment equation and arrives at a highly simplified linearized four degree of freedom equations for an inertially cross coupled aircraft performing a steady roll. The equations used by him are:

$$
\begin{aligned}
\dot{\alpha} &= q - P_{o}\beta \\
\dot{\beta} &= r + P_{o}\alpha \\
\dot{q} &= -P_{o}r(I_x - I_z)/I_y \\
\dot{r} &= -P_{o}q(I_y - I_x)/I_z \qquad (2.2.3)
\end{aligned}
$$

where P_o is the steady roll rate. He, like Phillips, Kolk and others, [8] investigates the stability of the aircraft

by applying Routh's test to the 4th degree characteristic equation. If an aircraft is unstable during a roll, the stability can be improved by increasing the "damping in yaw" and "damping in pitch". This of course can be achieved by increasing the areas of vertical and horizontal stabilizers respectively; but this calls for extensive modification to the aircraft and will be more expensive than incorporating an automatic control system. Moreover, the increased stabilizer area results in increased weight (the increase in weight being more than that of a control system) which affects the overall performance of the aircraft. Blakelock indicates how this increase can be effected by a control system incorporating yaw and pitch rate feedbacks only. But the tactical use of this control system is very limited, as during a combat, the fighter pilot is not interested in rolling continuously for more than 90° at the maximum. Of course this control system is useful, when one wants to perform more than two or three rolls, as during display flying.

Thus one can see that the stability aspect of cross coupling has received considerable attention in literature. But quite apart from instability, the problem of response for a stable motion may be most important for many aircraft during practical rolling maneuvers. In fact some aircraft may not be capable of rolling fast enough (relative to

their non-rolling natural frequencies) to get into the critical divergent region. Still they may experience severe and sometimes even hazardous response characteristics at roll rates below the critical value, as shown by Welch and Wilson [9]. They discuss the effect of various terms in the five degree freedom equations. They also develope an automatic control system for an ideal roll maneuver initiated by an abrupt aileron deflection. The ideal maneuver described by them is one in which side slip, side slip rate, change in angle of attack and angle of attack rate are all zero. In other words, the aircraft is desired to roll about the velocity vector. Expressions have been derived for ideal yaw and pitch controls. Problems in mechanization of this control law have also been discussed. Although this paper gives an elegant qualitative approach to the problem of automatic control during abrupt rolling maneuvers, unfortunately, the control law is neither optimal nor has it any tactical use.

2.3 EFFECT OF NEGLECTING INERTIAL COUPLING TERMS

With a view to arrive at a simplified model for the thesis problem, the effect of neglecting the inertial coupling terms on the performance of the aircraft was analysed for two types of control inputs both involving roll, (i) a pure aileron roll and (ii) a roll pullout maneuver with saturation type elevator control and a step

aileron input. For both these inputs, the five degree freedom model (2.2.2) was analysed for a swept wing fighter aircraft flying at 0.7 Mach at sea level. The various coefficients of the equations are given in Table 2.1 for the example aircraft for three flight conditions. The following three cases were analysed.

(1) The inertial coupling term in the pitching moment equation $(I_z - I_x)/I_y$ was neglected.

(2) The inertial terms in both pitching and yawing moment equations were neglected.

(3) All inertial terms were included.

In all the above cases, the resulting nonlinear system equations were solved on a digital computer and the plots of α, β, roll rate and normal acceleration were obtained (Figs. 2.1 to 2.4).

From the plots it can be seen that

(1) For a pure aileron roll,

(i) Inclusion of both inertial terms gives a well behaved response exhibiting interchange of α and β. The maximum roll rate is 241 deg/sec and the time taken to attain this roll rate is about 0.6 sec. β_{max} is 1.6° and α_{max} is 3° with a maximum normal acceleration of $3G^s$.

(ii) Inclusion of the coupling term in yawing moment equation $(I_x-I_y)/I_z$ alone results in a divergent

performance. The angle of attack and sideslip increase monotonically in the same direction, that after 4 seconds, the sideslip is 23.46° and angle of attack is 19.4° with a corresponding normal acceleration of 14G^{s}. The roll rate also keeps building up to a high value of about 391.7°/sec in 3.8 seconds.

(iii) Omission of both the coupling terms and considering a linear model results in a response, which though bounded, is far from the actual response with the inertial terms included. The maximum values are $\alpha_{max} = 2.3°$, $\beta_{max} = 3.3°$, maximum roll rate = 260°/sec and normal acceleration 2G^{s}. The maximum steady roll rate is obtained after 1.72 seconds.

A comparison of the plots leads to the conclusions.

(1) In a rolling maneuver, neglecting the effect of inertial coupling terms gives rise to a system whose both transient and steady state response are different from the actual system.

(2) The rolling time constant is larger when the coupling terms are omitted. It is nearly 3 times larger for the example aircraft.

(3) The actual side slip (with coupling terms) builds up first in the positive direction but then settles down to a negative steady state value. Neglecting the

inertial terms, completely distorts the response. The sideslip in this case never becomes negative; it settles down at a positive steady state value.

(4) The actual roll maneuver remains a positive G maneuver throughout. Though there are oscillations in G, it never becomes negative; but neglecting the coupling terms increases the amplitude of G oscillations and makes it negative during part of the cycle.

(5) The interchange of angles between sideslip and angle of attack is not maintained when the coupling terms are neglected.

(6) The steady state value of the roll rate is an optimistic figure when the coupling terms are neglected.

Thus a linear model cannot satisfactorily represent an aircraft performing a roll maneuver. The term $pq\ (I_y - I_x)/I_z$ in the yawing moment equation is seen to be responsible for the divergent nature of the solution when only this term is included. The coupling term in the pitching moment equation seems to stabilize the divergence to a new steady state.

A comparison of the plots obtained for the roll pullout maneuver also reveals that there are marked differences in both transient and steady state responses

between the nonlinear and linear models; and many of the above conclusions arrived for an aileron roll are also applicable for this maneuver.

2.4 CONCLUSIONS

Inertial cross coupling plays a very important role in a modern fighter aircraft performing any maneuver involving rolling. Analysis of the system without including the inertial coupling terms is incomplete and provides erroneous response. In general, omission of these terms results in an underestimate of sideslip, angle of attack and normal acceleration and an overestimate of maximum/steady roll rate.

TABLE 2.1 : STABILITY DERIVATIVES FOR EXAMPLE AIRCRAFT

FLIGHT CONDITION			PARTICULARS	PARTICULARS	FLIGHT CONDITION		
C	B	A			A	B	C
− 15.190	− 20.910	− 9.990	L_β	MACH NO.	0.9	0.7	0.7
−146.400	−543.800	−684.400	$L_{\beta\alpha}$	ALTITUDE ft	20,000	SEALEVEL	30,000
198.000	1456.000	1358.000	$L_{\beta\alpha 2}$	SPEED ft/sec	933.300	781.200	696.500
− 0.024	0.013	0.002	N_p	Z_α	− 1.329	− 1.746	− 0.618
− 0.459	− 1.583	− 1.578	$N_{p\alpha}$	Y_β	− 0.196	− 0.280	− 0.098
1.165	3.160	6.062	$N_{p\alpha^2}$	L_p	− 3.933	− 5.786	− 2.101
− 19.600	− 60.270	− 45.830	$L_{\delta a}$	$(I_x-I_y)/I_z$	− 0.716	− 0.716	− 0.716
4.010	12.270	15.850	$L_{\delta a^2}$	$(I_y-I_z)/I_x$	− 0.727	− 0.727	− 0.695
22.300	64.600	63.500	$L_{\delta a\alpha}$	$(I_z-I_x)/I_y$	0.949	0.949	0.946
− 3.970	− 11.450	− 30.530	$L_{\delta a^2\alpha}$	M_α	−23.180	−10.700	− 5.860
− 0.215	− 1.282	− 0.921	$N_{\delta a}$	M_q	− 0.814	− 1.168	− 0.452
0.041	0.207	0.148	$N_{\delta a^2}$	$M_{\delta e}$	−28.370	−31.640	−11.550
1.075	2.459	1.132	$N_{\delta a\alpha}$	N_β	5.670	8.880	3.160
0.091	0.346	1.445	$N_{\delta a^2\alpha}$	N_r	− 0.235	− 0.377	− 0.136

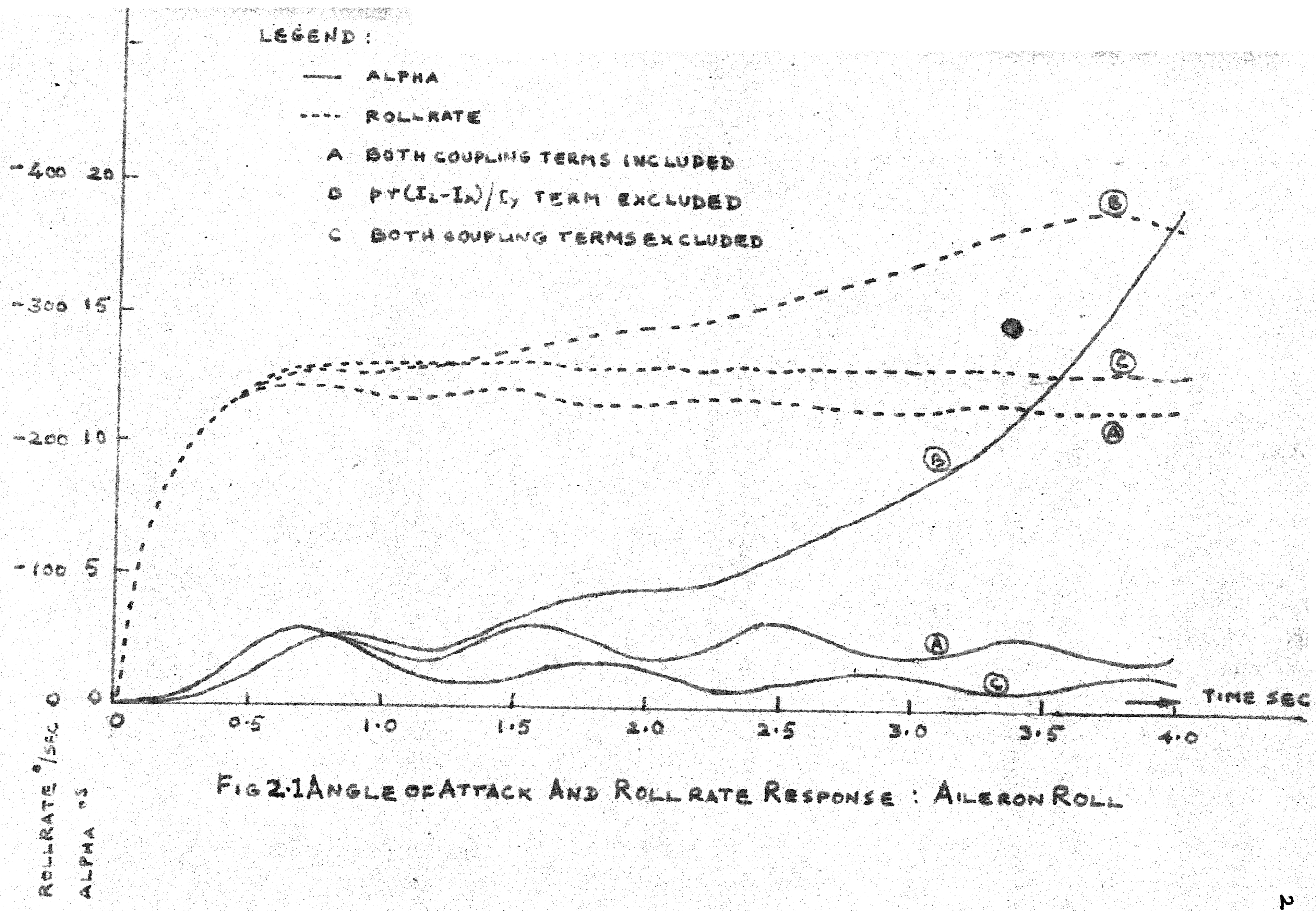

FIG 2.1 ANGLE OF ATTACK AND ROLL RATE RESPONSE : AILERON ROLL

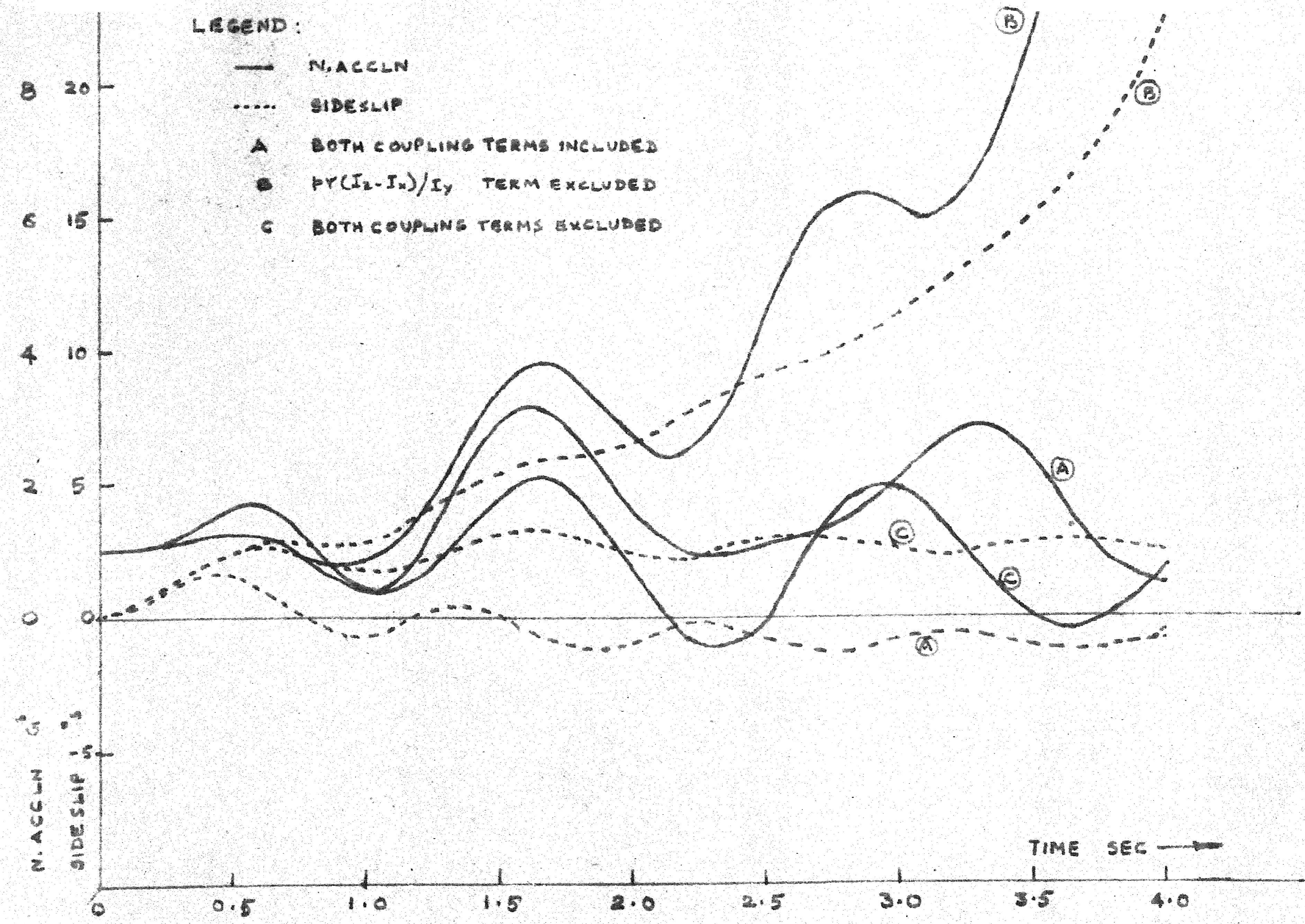

Fig 2.25 Sideslip And Normal Acceleration Response : Aileron Roll

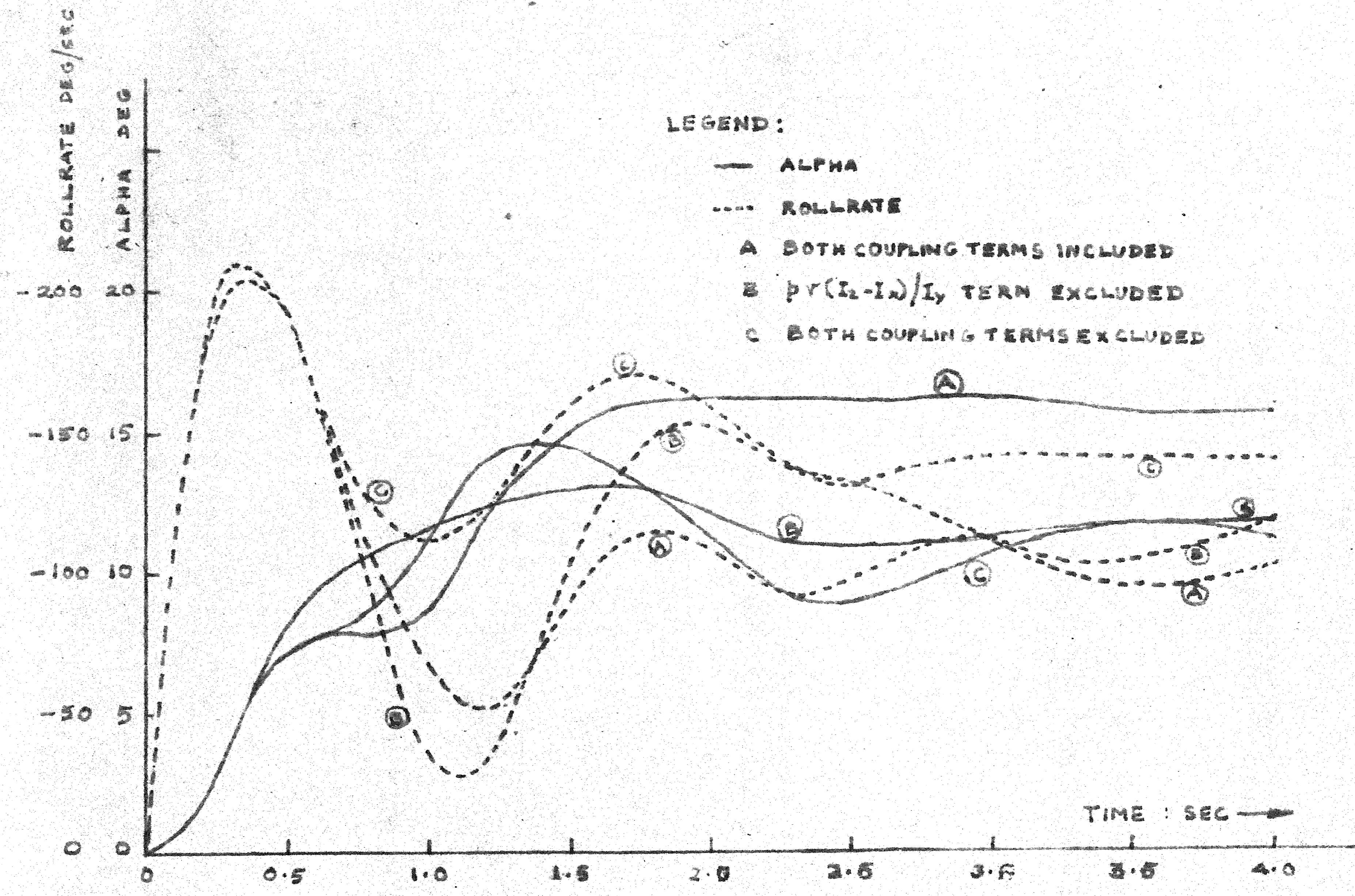

FIG 2.3 ANGLE OF ATTACK AND ROLLRATE RESPONSE : ROLL PULLOUT

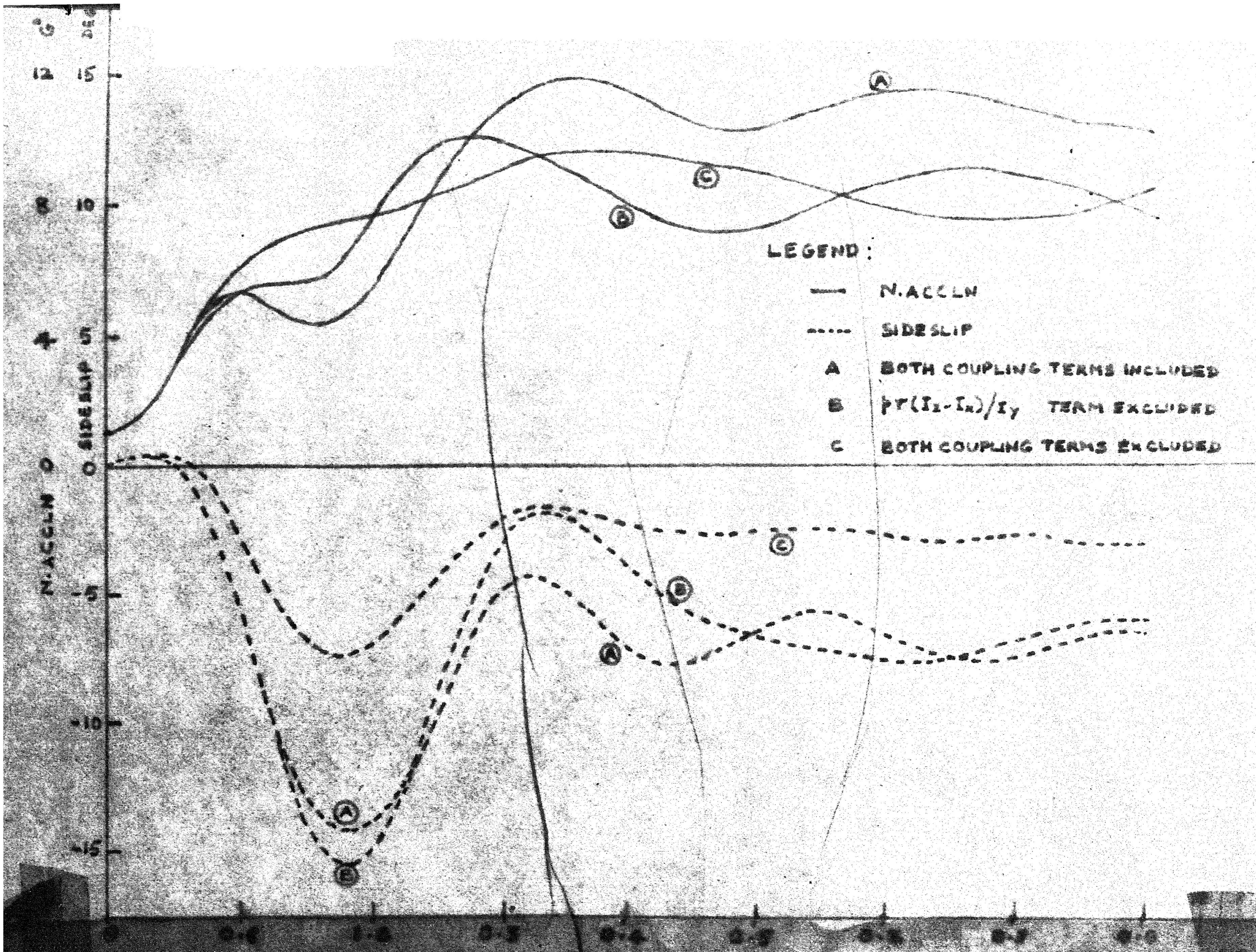

FIG 2.4 SIDESLIP AND NORMAL ACCELERATION RESPONSE: ROLL PULLOUT

CHAPTER III

PHYSICAL DESCRIPTION AND MATHEMATICAL MODEL FOR THE ROLLING PULLOUT MANEUVER

3.1 INTRODUCTION

In a mathematical study of any physical problem, it is necessary to arrive at a suitable mathematical model which incorporates as many physical constraints of the problem as is feasible, without introducing undue complexity. In this chapter, the physical requirements of the rolling pullout maneuver problem are clearly outlined. A suitable mathematical model is presented which will represent the physical system under all operating conditions encountered in this maneuver. The various assumptions and simplifications made in arriving at such a model are spelt out.

3.2 PHYSICAL REQUIREMENTS OF THE MANEUVER PROBLEM

Rolling pullout is one of those maneuvers which can be used in both offensive and defensive roles. The maneuver is usually performed from straight and level flight. At the end of this maneuver, the aircraft has rolled through about 80° and is subjected to high normal acceleration. This puts the aircraft in a tight turn, which is the object of the maneuver. The maneuver requires the operation of two control surfaces - the elevator and aileron, which can be performed in innumerable ways. The optimal way is defined

by the following two conditions :

(1) The time taken to roll through a desired roll angle must be a minimum.

(2) At the end of this time, the aircraft must be subjected to maximum normal acceleration.

These two conditions have been arrived at from tactical requirements of gaining combat superiority over the enemy. The thesis problem is to determine the optimal manner in which the two control surfaces are to be operated to perform this maneuver.

3.3 AN ACCURATE MODEL

This maneuver can broadly be classified under a large disturbance maneuver; hence the equations of motion based on small disturbance theory are no longer valid for this maneuver. Apart from the disturbance being large, there are other factors like inertial cross coupling etc. in a modern aircraft whose effects are significant in this particular maneuver and which call for the inclusion of a few terms which are normally neglected in the equations of motion.

Rhoads and O'hara [1] have conducted an extensive study of this maneuver in connection with estimation of tail loads encountered during this maneuver. They have proposed a six degree of freedom model for the aircraft

and have derived a set of dynamic equations to represent the roll pullout maneuver as completely as possible. The model includes many physical realities peculiar to this maneuver and which are normally omitted in the small disturbance linearized theory. They are :

(1) Changes in Mach No. are included through the speed equation.

(2) The products of the disturbances are considered significant; thus terms like wp, vr, qr etc. are included in the equations.

(3) The exact relations are used for obtaining the angular displacement about a set of fixed axes in terms of the angular velocities about the stability axes. This means, for instance, the pitch angle θ is not equal to $\int q \, dt$ as it is in small disturbance theory.

(4) Variations in stability derivatives due to angle of attack and Mach number changes are included. (These terms may vary from aircraft to aircraft) Also the aerodynamic and inertial cross coupling terms have been included.

However in their derivation the following few simplifying assumptions have been made to reduce the complexity of the equations :

(1) The effects of non-stationary flow are omitted. Previous work has shown that this will not cause significant error [12].

(2) Additional degrees of freedom due to aero-elastic effects are neglected. This is a valid assumption for any fighter aircraft which is structurally quite stiff.

(3) Engine gyroscopic effects are not included, as the moments introduced by them are normally of negligible magnitude.

The increase in accuracy obtained by inclusion of these terms is negligible as compared to the additional efforts required to solve a set of more complex equations.

3.4 A PRACTICAL WORKABLE MODEL

Though the model derived by Rhoads and O'hara [1] represents the aircraft accurately during the maneuver considered, the solution of these equations involve considerable time and effort even if the available machine techniques are made use of. As pointed out by Rhoads himself [10], these equations can be further simplified without highly jeopardising the accuracy of the results obtained. This is required so that the equations can be easily simulated and solved. Here, with the help of a few simplifying assumptions, a five degree of freedom model is arrived at.

In general, it is more convenient to express the equations of motion in terms of the disturbances from the initial conditions such that the effects of a particular control input may be readily apparent. The maneuver

investigated in this thesis is assumed to be initiated from a straight and level unaccelerated flight with rudder locked. A rudder locked case is considered since the maneuver is normally performed with aileron and elevator only. Under these conditions, many of the initial conditions reduce to zero. Extension of the method to take care of other initial conditions is indicated towards the end of the thesis. The following simplifying assumptions were made in arriving at the workable model :

(1) As the forward speed change is not appreciable during the maneuver, especially when the maneuver completes within one second, the speed equation was eliminated. This makes a five degree of freedom model for the aircraft; and eliminates the terms involving Mach number variations and hence greatly simplifies the equations of motion.

(2) $\sin\theta = \tan\theta = \theta$ and $\cos\theta = 1$. Even with a maximum value of 10 degrees for θ, the maximum error would be less than ± 1 percent with this assumption. However, it must be emphasized that this would not necessarily be true if the maneuver were started from diving or climbing conditions. Also in this thesis, wherever possible, this assumption is not made use of.

(3) $\beta = v/u$ and $\alpha = w/u$. This follows from the small angle assumption. Even otherwise, these assumptions can be validated in an optimal control problem by suitable constraints on β and α as we shall see in the next chapter.

(4) A few aerodynamic cross coupling terms which do not contribute significantly to the performance, have been omitted.

(5) Gravitational forces are neglected as they are normally insignificant for the speed range considered.

(6) $L_{\delta a}$ and $M_{\delta a}$ are adjusted for the usual decrease in aileron effectiveness at large deflections.

(7) The equations are written with respect to principal axes and hence the product of inertia terms like I_{xz} drop out automatically.

(8) The air speed is high enough so that the wing angle of attack in level flight is small.

With the above assumptions, equations (2.2.2) were arrived at which are reproduced here for convenience.

$$\begin{aligned}
\dot{\alpha} &= Z_\alpha \alpha - p\beta + q \\
\dot{\beta} &= p\alpha + Y_\beta \beta - r \\
\dot{p} &= L_\beta(\alpha)\beta + L_p p + qr\,(I_y - I_z)/I_x + L_{\delta a}(\alpha)\delta_a \qquad (3.4.1) \\
\dot{q} &= M_\alpha \alpha + M_q q + pr\,(I_z - I_x)/I_y + M_{\delta e}\,\delta e \\
\dot{r} &= N_\beta \beta + N_r r + N_p(\alpha)p + N_{\delta a}(\alpha)\delta a + pq\,(I_x - I_y)/I_z
\end{aligned}$$

where the nonlinear derivatives as functions of α are given below. These functions will vary from aircraft to aircraft and can be developed with the help of the performance curves and wind tunnel test results. The

functions given below are for a particular swept wing fighter.

$$L_\beta(\alpha) = L_\beta + L_{\beta\alpha}\alpha + L_{\beta\alpha^2}|\alpha|\alpha$$
$$N_p(\alpha) = N_p + N_{p\alpha}\alpha + N_{p\alpha^2}|\alpha|\alpha$$
$$L_{\delta a}(\alpha) = L_{\delta a} + L_{\delta a^2}|\delta a| + \alpha(L_{\delta a\alpha} + L_{\delta a^2\alpha}|\delta a|) \qquad (3.4.2)$$
$$N_{\delta a}(\alpha) = N_{\delta a} + N_{\delta a^2}|\delta a| + \alpha(N_{\delta a\alpha} + N_{\delta a^2\alpha}|\delta a|)$$

The following equations express the orientation at any time of the stability axes with respect to the space axes, in terms of the angular velocities p,q,r about the stability axes. These angles are marked in Fig. A-1 in Appendix A.

$$d\phi/dt = p + q \sin\phi \tan\theta + r \cos\phi \tan\theta$$
$$d\theta/dt = q \cos\phi - r \sin\phi \qquad (3.4.3)$$
$$d\psi/dt = (q \sin\phi + r \cos\phi)/\cos\theta$$

3.5 CONCLUSIONS

The above equations (3.4.1) to (3.4.3) represent a five degree freedom model for the maneuver and is a reasonably accurate one. The equations compare with those of Welch and Wilson also [9]. However, it must be pointed out that the model is not uniformly accurate for all flight conditions. For example, maneuvers performed at slightly greater than Mach 1, will result in deceleration through transonic region. The aerodynamic nonlinearities so introduced may have very large effects in rolling maneuvers and for such cases the original six degrees of freedom equations must be used.

CHAPTER IV

OPTIMAL CONTROL PROBLEM

4.1 INTRODUCTION

Although the aircraft control problems have so far been customarily tackled by the traditional transfer function methods, application of the Modern Control techniques results in several distinct advantages. For a maneuver problem of the kind described in this thesis, it becomes feasible to precisely specify the required performance and take into account the various physical constraints that are present. The resulting performance with the use of the control law so determined, is optimum in the sense that it cannot be improved upon any further without altering the problem itself. Hence the maneuver problem is first transformed into an optimal control problem with the definition of a state model and the formulation of suitable performance index. The physical significance of the constraints on the state and the control variables is brought out clearly. The gradient method of solving the optimal control problem is shown towards the end.

4.2 STATE MODEL

The system dynamic equations given in (3.4.1) can be transformed to the state equations by defining the following state and control variables.

(1) State Variables

x_1 = Angle of attack α in radians

x_2 = Side slip angle β in radians

x_3 = Rate of roll p in radians/sec.

x_4 = Rate of pitch q in radians/sec.

x_5 = Rate of yaw r in radians/sec.

x_6 = Roll angle ϕ in radians

x_7 = Pitch angle θ in radians

(2) Control Variables

Since a rudder locked case is considered, the only two control inputs are those due to elevator and aileron.

$\delta e(t)$ = elevator deflection in radians at time t.

$\delta a(t)$ = aileron deflection in radians at time t.

Control variables $u_1(t)$ and $u_2(t)$ are defined such that

$$u_1(t) = \delta e(t)/E_{max}$$

$$u_2(t) = \delta a(t)/A_{max}$$

where E_{max} and A_{max} are maximum possible elevator and aileron deflections respectively.

With the above definitions of the state and the control variables the state equations can be written as follows :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \end{bmatrix} = \begin{bmatrix} Z_\alpha x_1 - x_2 x_3 + x_4 \\ x_1 x_3 + Y_\beta x_2 - x_5 \\ L_\beta(x_1) x_2 + L_p x_3 + x_4 x_5 (I_y - I_z)/I_x \\ M_\alpha x_1 + M_q x_4 + x_3 x_5 (I_z - I_x)/I_y \\ N_\beta x_2 + N_r x_5 + N_p(x_1) x_3 + x_3 x_4 (I_x - I_y)/I_z \\ x_3 + x_4 x_7 \operatorname{Sin} x_6 + x_5 x_7 \operatorname{Cos} x_6 \\ x_4 \operatorname{Cos} x_6 - x_5 \operatorname{Sin} x_6 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & L_{\delta a}(x_1) \cdot A_{max} \\ M_{\delta e} \cdot E_{max} & 0 \\ 0 & N_{\delta a}(x_1) A_{max} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

(4.2.1)

where $L_\beta(x_1) = L_\beta + L_{\beta\alpha} x_1 + L_{\beta\alpha 2} |x_1| x_1$

$$N_p(x_1) = N_p + N_{p\alpha} x_1 + N_{p\alpha 2} |x_1| x_1$$

$$L_{\delta a}(x_1) = L_{\delta a} + L_{\delta a 2} |u_2| + (L_{\delta a\alpha} + L_{\delta a 2\alpha} |u_2|) x_1$$

$$N_{\delta a}(x_1) = N_{\delta a} + N_{\delta a 2} |u_2| + (N_{\delta a\alpha} + N_{\delta a 2\alpha} |u_2|) x_1$$

These are the nonlinear coefficients dependent on the angle of attack x_1 and the aileron deflection u_2. It may be observed that equation (4.2.1) is of the form

$$\dot{\underline{X}} = \underline{f}(\underline{X}, u) + B(\underline{X}, \underline{u})\, \underline{u} \qquad (4.2.2)$$

where $\underline{X}^T = [x_1\ x_2\ \ldots\ x_7]$

4.3 PERFORMANCE INDEX

The problem now is to find an optimal control which will perform the given maneuver in the shortest possible time while simultaneously achieving maximum normal acceleration at the end of the maneuver. Hence the problem calls for

optimization of two criteria simultaneously. However, this bi-criterion problem can be reduced to an equivalent scalar criterion problem, by associating a suitable weighting factor with one of the two criteria. Thus the scalar performance index may be defined as

$$J'' = W\left[-A\left\{\underline{X}(t_f)\right\}\right] + \int_{t_0}^{t_f} dt \qquad (4.3.1)$$

where $A\left\{\underline{X}(t_f)\right\}$ = Normal acceleration at terminal time. The expression for the normal acceleration at time t is

$$a_z(t) = V(\dot{\alpha} - q + \beta p)/g - \cos\theta \cos\phi \text{ in G units} \qquad (4.3.2)$$

The negative sign is introduced before the function A since it is a quantity to be maximized. W is the weightage factor (on normal acceleration at terminal time) which converts the bi-criterion problem to a scalar criterion problem. This has been introduced, since the scalar index consists of two dissimilar terms (one being time and the other, the normal acceleration). The weightage factor must be chosen to give a desired relative importance between time and normal acceleration at terminal time. A larger value for W results in relatively less importance being assigned to normal acceleration over the transition time. This is due to the fact that the normal acceleration is a quantity to be maximized, whereas the transition time is a quantity to be minimized. The effect of W on the optimal values of the transition time and the terminal

normal acceleration are shown for the example aircraft in Chapter V for a particular flight condition.

4.4 CONSTRAINTS

The performance index (4.3.1) is to be minimized subject to the undermentioned constraint equations and inequalities.

(1) Dynamic Constraints

The system state equations (4.2.1) form a set of dynamic differential constraint equations which must be satisfied along the entire optimal trajectory. These equations are to be satisfied with appropriate initial conditions, to be discussed later.

(2) State Variable Constraints

Since the problem pertains to the motion of an aircraft in air, the optimal trajectory is obviously confined to a particular region of the state space. This region is defined by the constraints on the various states as given below :

(i) x_1 - Angle of Attack

The normal acceleration of an aircraft is proportional to the angle of attack to a first order approximation. Hence maximizing normal acceleration implies an increase in angle of attack with time. Thus, for the maneuver initiated from a straight and level flight with a positive or zero angle of attack, x_1 should not become

negative during the maneuver. Also from physical consideration, each aircraft has got a maximum permissible angle of attack known as the 'stalling angle'. The upper bound on x_1 is therefore the 'stall angle' of the particular aircraft (α_{max}). The bounds on x_1 can be expressed by the inequality,

$$0 \leqslant x_1(t) \leqslant \alpha_{max}$$

This can be simplified and written as,

$$| 2x_1(t)/\alpha_{max} - 1 | - 1 \leqslant 0 \tag{4.4.1}$$

(ii) x_2 - Side Slip Angle

The side slip during the maneuver should not exceed the maximum tolerable side slip angle (β_{max}). This can be written as

$$| x_2(t)/\beta_{max} | - 1 \leqslant 0 \tag{4.4.2}$$

(iii) x_3 - Roll Rate

Though the objective is to attain the desired roll angle in minimum time, the roll rate should not become too high to cause disorientation to the pilot. A comfortably tolerable maximum roll rate may be fixed at 250 deg/sec (p_{max}). The corresponding constraint on x_3 is then

$$| x_3(t)/p_{max} | - 1 \leqslant 0 \tag{4.4.3}$$

(iv) x_5 - Yaw Rate

The yaw rate must be of the same sign as roll rate; as otherwise there will be adverse yaw which is not desired.

This is taken care of by the following inequality,

$$- x_5(t)\, x_3(t) \leqslant 0 \qquad (4.4.4)$$

(v) x_7 - Pitch Angle

The pitch angle must be positive through out, for the maneuver considered; and must be constrained to within about 10 degrees (θ_{max}) so that the small angle approximation of $\theta = \operatorname{Sin}\theta = \operatorname{Tan}\theta$ is valid. Hence

$$0 \leqslant x_7(t) \leqslant \theta_{max}$$

$$\text{or } \left| \, 2x_7(t)/\theta_{max} - 1 \right| - 1 \leqslant 0 \qquad (4.4.5)$$

(vi) $a_z(t)$ - Normal Acceleration

Being a positive G maneuver, the normal acceleration should not become negative during any part of the maneuver. The upper bound on a_z depends on one of the two following factors.

(1) The structural strength of the aircraft to withstand G.

(2) The pilot's ability to withstand G without getting blacked out.

The lower value of the above two decides the upper bound on a_z. However, in modern fighter aircraft, which are quite strong structurally, the second factor usually decides the maximum normal acceleration; and for a short duration maneuver of this kind, an acceleration of 7 to 8 G (a_{zmax}) can be laid as the maximum the pilot can withstand.

Hence the following constraint on x_8.

$$|2a_z(t)/a_{zmax} - 1| - 1 \leq 0 \tag{4.4.6}$$

(3) Control Variable Constraints

The only constraint on the normalized control variables u_1 and u_2 is, that their magnitudes must not exceed unity. This is obvious from the definitions of u_1 and u_2. Hence the constraints

$$|u_1| - 1 \leq 0 \tag{4.4.6}$$

$$|u_2| - 1 \leq 0 \tag{4.4.7}$$

4.5 INITIAL CONDITIONS

The control problem is solved for the particular case of the maneuver initiated from a straight and level flight. Hence,

$$\underline{X}(t_o) = \underline{0} \tag{4.5.1}$$

This forms the initial condition for solving the differential constraint equations (4.2.1) as also the initial manifold from where the optimal state trajectory starts.

4.6 TERMINAL TIME

Since time during which the controls are applied is a quantity to be minimized, the terminal time t_f is free. Also, at the terminal time, the performance index is minimum and its rate of change with respect to time is zero. Thus t_f can be found by equating dJ''/dt_f to zero.

4.7 TERMINAL MANIFOLD

The performance index (4.3.1) is to be minimized subject to the constraints of Section 4.4 such that at the terminal time given by (4.6) above, the optimal trajectory lies on the following terminal manifold.

$$
\begin{aligned}
x_2(t_f) &= 0 \\
x_3(t_f) &= 0 \\
x_6(t_f) &= \phi_d
\end{aligned}
\tag{4.7.1}
$$

where ϕ_d is the roll angle desired at the end of the maneuver. The other states are free at the terminal time subject to the state variable constraints which are to be satisfied at the terminal time as well.

4.8 OPTIMAL CONTROL PROBLEM

The optimal control problem can now be stated as follows :

Find an optimal control law $\underline{u}^*(\underline{X}, t)$ which minimizes the performance index 4.3.1 subject to the constraints of Section 4.4 and simultaneously transfers the system described by (4.2.1) from the initial manifold specified by (4.5.1) to the terminal manifold given by (4.7.1).

As a first step towards solving the problem, the constraints on state and control variables are absorbed in the performance index itself by modifying the latter with suitable penalty functions [13]. Thus the constraints on

$u_1(t)$ and $u_2(t)$ are taken care of by the addition of the following terms to the cost function within the integral sign

$$\sum_{i=1}^{2} G_{i,i}(u_i^2 - 1)\, E_i(1 - u_i^2) \tag{4.8.1}$$

where $G_{1,1}$ and $G_{2,2}$ are arbitrarily large positive numbers called penalties; and $E_i(1 - u_i^2)$ is the Heaviside step function with $(1 - u_i^2)$ as its argument such that

$$\left.\begin{aligned} E_i(1 - u_i^2) &= 0 \\ &= 1 \end{aligned}\right\} \begin{aligned} &\text{if } (1 - u_i^2) \geqslant 0 \\ &\text{if } (1 - u_i^2) < 0 \end{aligned}$$

$$i = 1, 2 \tag{4.8.2}$$

The constraints on the state variables are taken care of by defining additional state variables [14]. The inequality constraints on the state variables are transformed to differential constraint equations as shown below and added to the state equations (4.2.1)

$$\begin{aligned}
\dot{x}_8 &= \left\{(2x_1/\alpha_{max} - 1)^2 - 1\right\} D \left\{(2x_1/\alpha_{max} - 1)^2 - 1\right\} \\
\dot{x}_9 &= \left\{(x_2/\beta_{max})^2 - 1\right\} D \left\{(x_2/\beta_{max})^2 - 1\right\} \\
\dot{x}_{10} &= \left\{(x_3/p_{max})^2 - 1\right\} D \left\{(x_3/p_{max})^2 - 1\right\} \\
\dot{x}_{11} &= x_5^2 \; D \left\{- x_5 x_3\right\} \\
\dot{x}_{12} &= \left\{(2x_7/\theta_{max} - 1)^2 - 1\right\} D \left\{(2x_7/\theta_{max} - 1)^2 - 1\right\} \\
\dot{x}_{13} &= \left\{(2a_z/a_{zmax} - 1)^2 - 1\right\} D \left\{(2a_z/a_{zmax} - 1)^2 - 1\right\}
\end{aligned} \tag{4.8.3}$$

where $$D\left\{\cdot\right\} = \begin{cases} +1 & \text{if } \left\{\cdot\right\} > 0 \\ 0 & \text{if } \left\{\cdot\right\} \leq 0 \end{cases} \tag{4.8.4}$$

With the addition of these six equations to the original seven state equations, the resulting set can be written in the form

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) + \underline{B}(\underline{x}, \underline{u})\, \underline{u} \tag{4.8.5}$$

where

$$\underline{x}^T = \left[x_1, x_2 \ldots x_{13}\right] \tag{4.8.6}$$

The initial conditions of the new states (x_8 to x_{13}) are zero; and on the optimal trajectory, these states are zero again at the terminal time. Thus the initial and terminal manifolds are defined for all the 13 states.

Any deviation of the terminal states from the terminal manifold is an error and this error must be least. This is achieved by the addition of weighted error square terms to the performance index. Minimization of the performance index hence implies an automatic minimization of these errors.

The performance index after the inclusion of the weighted error square terms and the control variable constraints is given below

$$J = \frac{1}{2} S_{2,2}(x_{2f} - x_{2d})^2 + \frac{1}{2} S_{3,3}(x_{3f} - x_{3d})^2$$
$$+ \frac{1}{2} S_{6,6}(x_{6f} - x_{6d})^2 + \frac{1}{2} \sum_{i=8}^{13} S_{i,i}(x_{if} - x_{id})^2 - Wa_z(t_f)$$
$$+ \int_{t_o}^{t_f} \left\{ 1 + \frac{1}{2} \sum_{i=1}^{2} G_{i,i}(u_i^2 - 1)E_i(1 - u_i^2) \right\} dt \qquad (4.8.7)$$

where the subscript 'f' for the state denotes the actual value of that state at terminal time and subscript 'd' for the desired value. $S_{i,i}$ and $G_{i,i}$ are arbitrarily large positive numbers.

With the above simplifications, the optimal control problem reduces to that of finding an optimal control $\underline{u}^*(\underline{x}, t)$ which minimizes the performance index J subject to the differential constraints (4.8.5) and which simultaneously transfers the system from $\underline{X} = \underline{0}$ to the desired terminal manifold. Thus the problem with the constraints can be identified as a Bolza type problem.

4.9 SOLUTION BY GRADIENT TECHNIQUE

A closed form analytical solution to a Bolza type problem is obtainable only in the special situation when the constraints are linear. The differential constraint equations in our problem being nonlinear, a closed form solution is not feasible and hence we take recourse to numerical methods. Of the various numerical methods that are available, the 'Gradient Method' is chosen because it possesses some attractive features :

(1) Convergence is very fast in the beginning. This is more suitable for the problem on hand, where very high degree of accuracy is not required; hence results in a saving in computation time.

(2) It is very easy to generate an algorithm for a problem of this complexity.

(3) As the nature of the solution is known before hand, it is easier to make a reasonably good guess of the initial control vector.

A brief outline of the method as applied to the thesis problem is given below :

(1) Applying Pontriagyn's Maximum principle, a Hamiltonian H is defined as

$$H(\underline{x},\underline{u},\underline{\lambda},t) = \underline{\lambda}^T[\ \underline{f}(\underline{x},\underline{u}) + \underline{B}(\underline{x},\underline{u})\underline{u}\ ] + \phi(\underline{x},\underline{u},t) \qquad (4.9.1)$$

where $\underline{\lambda}^T = [\lambda_1, \lambda_2 \ldots \lambda_{13}]$ are the costates and

$$\phi(\underline{x},\underline{u},t) = 1 + \frac{1}{2} \sum_{i=1}^{2} G_{i,i}(u_i^2 - 1)\ E_i(1 - u_i^2) \qquad (4.9.2)$$

(2) The costate equations are written by differentiating the Hamiltonian partially with respect to the state vector.

$$\dot{\underline{\lambda}} = -\partial H/\partial \underline{x} \qquad (4.9.3)$$

The boundary conditions for the above costate equations at terminal time are given by

$$\underline{\lambda}(t_f) = \partial\ \Theta\ /\partial \underline{x}\ |_{t_f} \qquad (4.9.4)$$

where $\Phi = \frac{1}{2} S_{2,2}(x_{2f} - x_{2d})^2 + \frac{1}{2} S_{3,3}(x_{3f} - x_{3d})^2$

$$+ \frac{1}{2} S_{6,6}(x_{6f} - x_{6d})^2 + \frac{1}{2} \sum_{i=8}^{13} S_{i,i}(x_{if} - x_{id})^2$$

$$- Wa_z(t_f)$$

(3) An arbitrarily small step size 'h' is chosen for integration. A trial value for the optimal control vector $\underline{u}^N(t)$ is assumed. (This is arrived from previous experience). The state equations (4.8.5) are integrated in steps, in forward time using one of the numerical integration techniques; the initial conditions being $\underline{X} = \underline{0}$. At each step of integration, dJ/dt_f is evaluated and the integration is stopped when $dJ/dt_f = 0$. The time t_f at which this condition is met, gives an approximate estimate of the final time. The solution state vector is stored corresponding to the points of integration.

(4) Since $\underline{X}^N(t_f)$ has been computed in step 3, $\underline{\lambda}^N(t_f)$ can be computed from (4.9.4). Now the costate equations (4.9.3) are integrated in reverse time with the boundary conditions at the terminal time being given by (4.9.4).

(5) The components of the gradient vector $\partial H/\partial u_1$ and $\partial H/\partial u_2$ are computed from

$$\partial H/\partial u_1 = {}_4M_{\delta e}E_{max} + G_{1,1}u_1E(u_1^2 - 1) \qquad (4.9.5)$$

$$\partial H/\partial u_2 = \lambda_3 \left[L_{\delta a} + 2L_{\delta a^2} |u_2| + (L_{\delta a \alpha} + 2L_{\delta a^2 \alpha} |u_2|) x_1 \right] A_{max}$$
$$+ \lambda_5 \left[N_{\delta a} + 2N_{\delta a^2} |u_2| + (N_{\delta a \alpha} + 2N_{\delta a^2 \alpha} |u_2|) x_1 \right] A_{max}$$
$$+ G_{2,2} u_2 E(u_2^2 - 1) \qquad (4.9.6)$$

The gradient vector $\partial H/\partial \underline{u}$ will be null only when the conditions for a minimum are satisfied. The change in the performance index at N^{th} iteration to a first order approximation is

$$\Delta J^N = \left[\partial H/\partial \underline{u}^N \right]^T \Delta \underline{u}^N \qquad (4.9.7)$$

where $\Delta \underline{u}^N$ is the change in the control vector at the N^{th} iteration. If we wish to make the greatest or steepest change ΔJ^N in J, this can be effected by making $\Delta \underline{u}^N$ directed opposite to the gradient so that the dot product of $\partial H/\partial \underline{u}^N$ and $\Delta \underline{u}^N$ is maximum.

$$\text{(i.e.)} \quad \Delta \underline{u}^N = -K \left[\partial H/\partial \underline{u}^N \right]$$
$$\text{or} \quad \Delta J^N = -K \left[\partial H/\partial \underline{u}^N \right]^T \left[\partial H/\partial \underline{u}^N \right] \qquad (4.9.8)$$

where K is an arbitrary positive constant less than unity. Usually K is chosen to give a 10 percent change in the value of the performance index.

For a 10 percent change in J, $K = 0.1 [J^N] / \langle \partial H/\partial \underline{u}^N, \partial H/\partial \underline{u}^N \rangle$ where $\langle \, . \, \rangle$ denotes scalar product.

(6) The new trial value of the control for the $N+1^{th}$ iteration, $\underline{u}^{N+1}$, is given by

$$\underline{u}^{N+1} = \underline{u}^N + \Delta \underline{u}^N \qquad (4.9.9)$$

(7) With this new control, $\underline{x}^{N+1}$ and $\underline{\lambda}^{N+1}$ are computed as before and the iterative process repeated until the change in the cost function $(\Delta J)^N$ between two successive iterations is below some specified small value.

(8) Under J minimum condition indicated by negligible change in J, the terminal manifold equations will also be satisfied to a reasonable degree of accuracy. The accuracy is increased by increasing the appropriate penalty $S_{i,i}$ and repeating steps 1 to 7 once again with the optimum control obtained in the first stage.

An example problem is solved in Chapter V for a modern fighter aircraft.

4.10 CONCLUSIONS

The maneuver problem is formulated as an optimal control problem incorporating all the physical constraints. The problem is solved for a swept wing fighter aircraft; and some computational aspects of solving the problem by Gradient method are discussed in the next chapter.

CHAPTER V

SOME COMPUTATIONAL ASPECTS AND RESULTS

5.1 INTRODUCTION

The special features of the gradient method as applied to the optimal maneuver problem are discussed here in greater detail. A block diagram of the computer flow chart and the functions of the various subroutines are given in Figure 5.1. An example problem is solved for different flight conditions and different values of W and the optimal control and trajectories are plotted and thoroughly discussed. Extension to other initial conditions and implementation aspect of the optimal control are also presented.

5.2 CERTAIN COMPUTATIONAL ASPECTS

The following aspects of computation need special consideration.

(1) Initial choice of control

Since this is a nonlinear problem solved by the gradient method, the control vector $\underline{u}$ will be obtained as a function of time. Later on, after the optimal controls have been computed for different flight conditions, one may attempt to solve the so called Synthesis problem by

expressing optimal control as a function of the state and time. Hence, to start with, the control vector was arrived at for the example problem by finding the response of the aircraft to three different types of probable control inputs in performing this maneuver :

(i) A negative step aileron and a step elevator applied simultaneously.

(ii) A negative step aileron and a ramp type elevator saturating at the end of 1 second.

(iii) A negative step aileron; and a step elevator applied 0.4 second after aileron is applied.

From the roll rate, roll angle, side slip and normal acceleration responses for the above three cases, it was decided to choose, a negative step aileron for 0.4 second followed by a positive going ramp type aileron saturating at the end of 0.8 second; and a ramp elevator saturating in 0.1 second. This type of control input is expected to provide a response in between those of (i) and (ii) of the trial inputs and will be close to the desired response.

(2) Integrating technique

The system equations were integrated by Adam's Fourth order Predictor corrector method. The solution at the first 4 points was obtained by Fourth order Runge Kutta method with Gill's variation. The Predictor corrector, though very accurate, was found to be highly sensitive to

the step size of integration for this problem. A larger step size and high accuracy resulted in oscillations during integration. This was predominant when the costate equations were integrated with large values for initial conditions. Since the basic problem is to find the integrated value at each integration point and not the final value alone, automatic reduction of step size as the integration proceeds, cannot be resorted to. On the other hand reducing the step size to arbitrarily small value calls for increase in computer memory since the solution at the integration points are to be stored for calculating the gradient, performance index etc. However, this difficulty was overcome by a few trial runs with various step sizes and ultimately a step size of 0.0025 sec was decided upon with 1 percent accuracy for the example problem.

(3) Terminal time

Since the terminal time is not fixed, the forward integration of state equations was stopped when the total time derivative of the performance index (dJ/dt_f) passed through zero. This is only a necessary condition. The sufficient condition, namely the second derivative must be positive was not tested due to complexity of the expressions involved and consequent increase in computation time. This gave rise to stopping the integration at incorrect times. (The integration stops when dJ/dt_f is zero for the first

time as the integration proceeds; but this need not necessarily be the final time at which many other conditions are to be satisfied.) These incorrect points were to be eliminated by examining the terminal states at these points which will be far away from the desired terminal manifold; and also the controls will be operating outside the boundary. In the thesis, the algorithm has been developed in such a way that the test for dJ/dt_f to be zero is done only when the states are in the viscinity of the terminal manifold. This avoids testing for $dJ/dt_f = 0$ and $d^2J/dt_f^2 \geqslant 0$ at all integration points and hence saving considerable computer time.

(4) Step size down the gradient

The gradient method as applied to this problem is highly sensitive to the step size K which decides the change in control required for the next iteration. Too large a step size may result in an increase in performance index instead of a decrease. Too small a step size on the other hand may increase the number of iterations before satisfactory convergence is obtained. In the example problem, the step size was calculated for 10 percent change in the value of the performance index; the control vector was modified accordingly and the system equations were integrated and the new value of the performance index was evaluated. If this new value is less than the previous one, the process is continued, till such time the value of the index starts

increasing. At this stage the step size is reduced to 1/10 of its calculated value (for 10 percent change) and the process repeated.

(5) Penalties

The constraints on control variables were very rarely violated unless the region of search for $dJ/dt_f = 0$ was incorrect. The constraints on state variables and terminal manifold were usually violated. This violation was gradually removed by increasing the penalties on those variables in stages. One stage of iterations was carried out with one set of penalties till there was insignificant change in the value of the performance index between successive iterations. Then the penalties increased and the whole iterative cycle started once again with the controls obtained from the previous stage. Thus the optimization has to be performed in stages; the penalties being increased at the beginning of each stage. Ultimately the penalties on a few states were increased to 10^9 for the example problem.

(6) Initial conditions

The initial conditions for solving the state equations remain constant throughout the process of optimization. But the initial conditions for the costate equations is proportional to the error in the violation of the state terminal manifold and are also proportional to the penalties on the states. Hence either large violation or large penalty

results in large values for the initial conditions. These large initial conditions resulted in instability during integration by Predictor corrector method. Hence it was absolutely necessary that as accurate a result as was possible be obtained in any stage with one set of penalties before the penalties were increased for the next stage.

5.3 FLOW CHART

A block diagram of the computer flow chart is given in Figure 5.1. The various subroutines used are :

(1) CONTRL

The initial choice of the control vector is either read through or generated in this subroutine. Various tolerances which require frequent change are also fed through this subroutine.

(2) SOLN

This subroutine integrates either state or costate equations (depending on the index value given by the main program) by calling subroutines PREDTX or PREDTL respectively, and stores the integrated result at each integration point. While solving state equations, change of the sign of dJ/dt_f is searched in the proper zone and integration is stopped when there is a change in sign.

(3) PREDTX and PREDTL

These are the subroutines where the state and the costate equations are written.

(4) DOTPRD

This subroutine finds the dot product of two vectors. This is used to find the dot product of the gradient vector with itself, for estimating the step size K down the gradient.

(5) MAX

The subroutine finds the maximum absolute value of a given array of real numbers. This is required to normalize the optimal state solution at the solution points before obtaining a plot.

(6) PLOT

This subroutine provides a normalized plot of the optimal solution. The plot is not very accurate since the normalized values are truncated before plotting; but gives a reasonably good idea about the solution.

5.4 EXAMPLE

The rolling pullout maneuver problem is solved for a swept wing fighter aircraft with the following leading particulars :

W_t/S = 65 lb/sq.ft.

Aspect ratio = 3.5

Sweep back = 45°

$(I_y - I_x)/I_z$ = 0.716

The stability derivatives for this aircraft were known for three different flight conditions. These derivatives were

given in Table 2.1. The optimal control and the optimal response were found for three different flight conditions with value of $W = 1$.

(A) 0.9 Mach at 20000 ft.

(B) 0.7 Mach at sea level

(C) 0.7 Mach at 30000 ft.

For flight condition A, the response and control were found for $W = 1000$ and $W = 0.0001$ also, to study the effect of the weightage factor W on the terminal time and normal acceleration at terminal time. The results are plotted. (Figures 5.2 to 5.6).

5.5 DISCUSSION OF THE RESULTS

Though it is impossible to generalize based on the results for only three flight conditions, some broad conclusions can be drawn and the general trend of the solution discussed. With these in view, a discussion of the optimal control and the resulting optimal trajectory and the conclusions drawn thereupon may be listed as follows :

(1) Transition time

The minimum time required to perform this maneuver is given in Table 5.1 for the three flight conditions. It is seen that for the same Mach number, the maneuver performed at a higher altitude (hence reduced true air speed) requires longer time. This is suggestive heuristically as well.

(2) Normal acceleration at terminal time

Normal acceleration decreases with increase in altitude. This is so because of the decrease in control effectiveness at high altitudes. For the same Mach number, the pilot flying low can maneuver his aircraft faster and can apply a higher value of G than while flying at higher altitudes.

(3) Nature of the optimal control

Since the maneuver calls for rolling fast to one side, a step aileron is suggested. Also since it is a stop to stop maneuver, (zero roll rate at the beginning and end of the maneuver) at some stage the aileron must be reversed and a negative aileron applied so that the roll rate becomes zero when the aircraft has rolled through the desired roll angle. This argument concurs with the type of optimal control obtained. As for the elevator, since the normal acceleration is required to be maximum only at the end, maximum elevator deflection must be present only towards the end. This suggests a saturation type elevator input. A study of the results shows that the nature of the optimal control obtained coincides very well with the nature of the control suggested heuristically.

(4) Constraints

It is seen that the constraint on β is violated to a certain extent towards the end of the transition time.

This violation can be reduced by a further increase in the penalty on that state; but since the violation was not very severe, this process was not carried on any further. Also the constraints on most of the states were fixed by pilot's opinion rating. This being highly subjective, violation of the constraints by a small percentage need not be viewed upon seriously.

(5) Maximum roll rate

A comparison of the roll rate plots for flight condition B without elevator (Figure 3.1) and the optimal roll rate shows that the maximum roll rate in the optimal case is less than that obtained for pure aileron roll, and the time taken to roll through 80° is more in the optimal case. This seems to question the optimality of the results obtained. But it must be borne in mind that the optimal problem was solved for a stop to stop maneuver, which calls for zero roll rate at the terminal time. In a pure aileron roll, the roll rate was approximately 80 percent of the maximum roll rate when 80° roll was executed.

(6) Effect of weightage factor W

It was mentioned in Chapter IV that the weightage factor W determines the relative importance in optimization between the two quantities (1) the transition time and (2) the normal acceleration. Analysis of the results reveals that for a given flight condition, the effect of W

is not very significant. For a change in W from 0.0001 to 1000, the corresponding change in transition time is only .005 seconds and the change in normal acceleration is 0.02G.

5.6 IMPLEMENTATION OF THE OPTIMAL CONTROL LAW

Fortunately the optimal control law for this problem is not highly complex. It can be easily implemented by suitable hardware. But before implementing, the problem must be solved for various combinations of Mach number and altitude and also for various initial conditions. The results obtained are to be compared and the optimal control must be expressed as a set of empirical relations in Mach number, altitude and initial conditions. These empirical relations can be transformed into a hardware realization using suitable logical circuits, amplifiers, actuators etc.

The maneuver is initiated by the pilot by an abrupt movement of the stick to one side which actuates a limit switch. Thereupon, depending on the conditions of flight, suitable control is applied by the logical circuits. The pilot takes over from the control system when the aircraft rolls through the desired roll angle.

5.7 EXTENSION TO OTHER INITIAL CONDITIONS

The maneuver problem was solved starting from an initial condition corresponding to straight and level unaccelerated flight. Although, the scope of this thesis

does not cover initial conditions other than straight and level flight, for other initial conditions, the original six degree of freedom equations of Rhoads and O'hara [1] can be used and the simplifying assumptions are to be made based on the characteristics of the initial conditions. For example if the maneuver is performed from a true banked turn in a horizontal plane, the initial side slip velocity is zero and the initial bank angle is a finite value. There will be finite initial pitching velocity which is balanced by the elevator and a constant yaw rate. Also all the three controls are deflected initially in a true banked turn and the rudder is assumed to be locked at this deflected position. These are some of the aspects to be borne in mind and the equations of motion should be modified accordingly.

5.8 CONCLUSIONS

Certain computational aspects peculiar to this problem are discussed and the computer flow chart and functions of various subroutines are given. An example problem in roll pullout maneuver is solved for a modern fighter aircraft whose stability derivatives were known for three different flight conditions. The resulting optimal control and trajectories are plotted. Conclusions based on the results obtained and implementation aspect of the control law are also presented.

TABLE 5.1 : RESULTS OF MANEUVER PROBLEM

FLIGHT CONDN.	MACH NUMBER	ALTITUDE FT.	W	TRANSITION TIME SEC.	NORMAL ACCLN G
A	0.9	20,000	0.0001	0.6950	4.9106
A	0.9	20,000	1	0.6925	4.9074
A	0.9	20,000	1000	0.6900	4.8906
B	0.7	SEA LEVEL	1	0.6800	8.2592
C	0.7	30,000	1	0.9600	1.8490

TABLE 5.2 : EFFECT OF 'W' ON OPTIMAL CONTROL

FLIGHT CONDITION 'A'

TIME SEC	ELEVATOR DEG.			AILERON DEG.		
	W=0.0001	W=1	W=1000	W=0.0001	W=1	W=1000
0.0000	0.0105	0.0024	0.0034	29.9954	29.9982	29.9965
0.0500	-1.2361	-0.4968	-1.2453	29.9918	29.9975	29.9953
0.1000	-2.4835	-2.4960	-2.4944	29.9881	29.9966	29.9937
0.1500	-3.7320	-3.7456	-3.7437	29.9847	29.9957	29.9921
0.2000	-4.9815	-4.9954	-4.9935	29.9819	29.9949	29.9906
0.2500	-4.9821	-4.9955	-4.9936	29.9800	29.9941	29.9894
0.3000	-4.9838	-4.9959	-4.9942	29.9795	29.9926	29.9884
0.3500	-4.9864	-4.9965	-4.9951	29.9675	29.9914	29.9875
0.4000	-4.9895	-4.9973	-4.9963	25.4778	25.4927	25.4848
0.4500	-4.9929	-4.9981	-4.9975	17.9798	17.9[illegible]23	17.9815
0.5000	-4.9958	-4.9989	-4.9986	10.4929	10.4944	10.4817
0.3500	-4.9981	-4.9995	-4.9994	3.0183	2.9994	2.9855
0.6000	-4.9994	-4.9998	-4.9999	- 4.4524	-4.4937	-4.5053
0.6500	-4.9999	-5.0000	-4.9999	-11.9339	-11.9882	-11.9960
0.6900	-5.0000	-5.0000	-5.0000	-17.9271	-17.9853	-17.9904
0.6925	-5.0000	-5.0000	-	-17.9260	-17.9851	-
0.6950	-5.0000	-	-	-18.6768	-	-

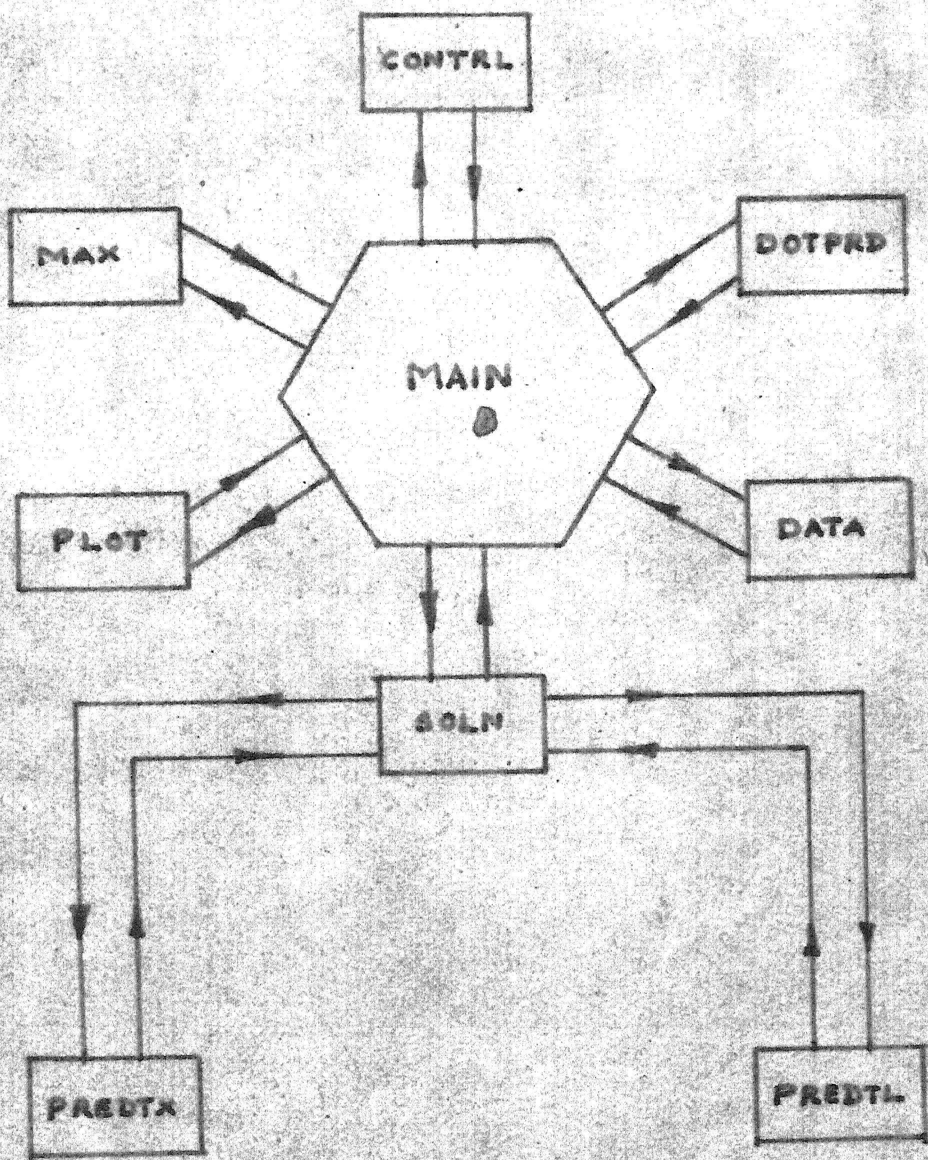

FIG 5.1 BLOCK DIAGRAM OF COMPUTER FLOW CHART

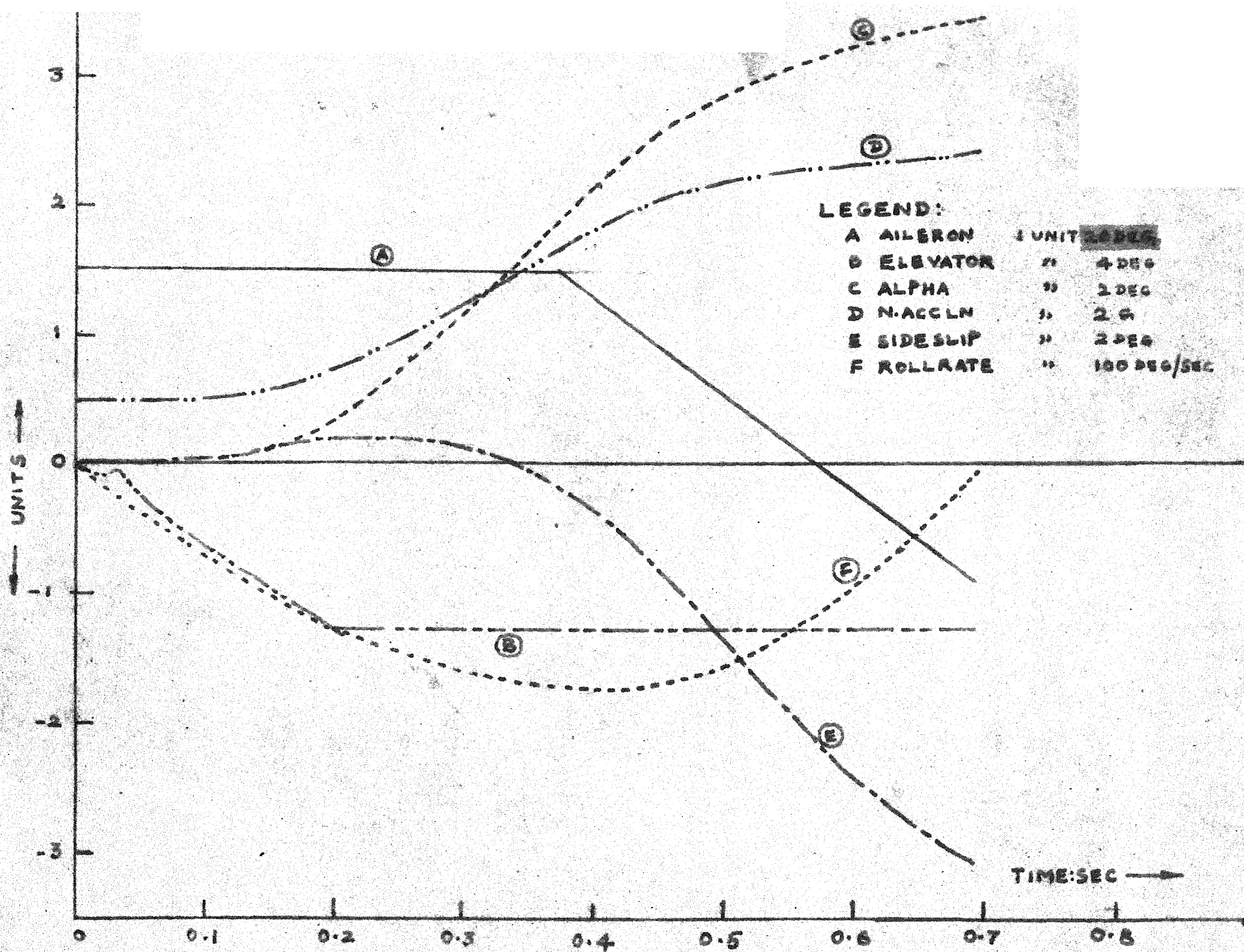

FIG 5.2 OPTIMAL CONTROLS AND TRAJECTORIES : FLIGHT CONDITION A

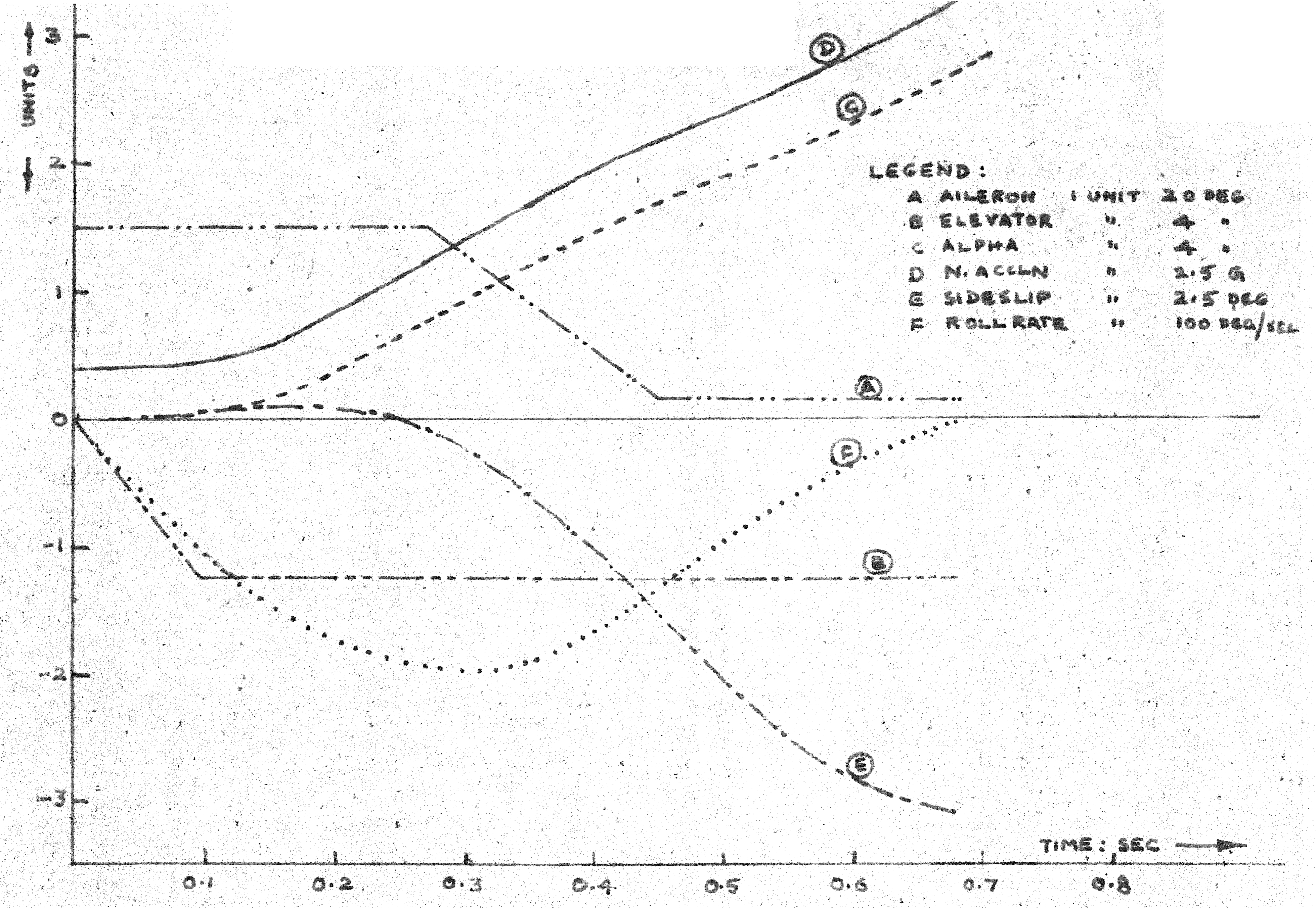

FIG 5.3 OPTIMAL CONTROLS AND TRAJECTORIES — FLIGHT CONDITION B

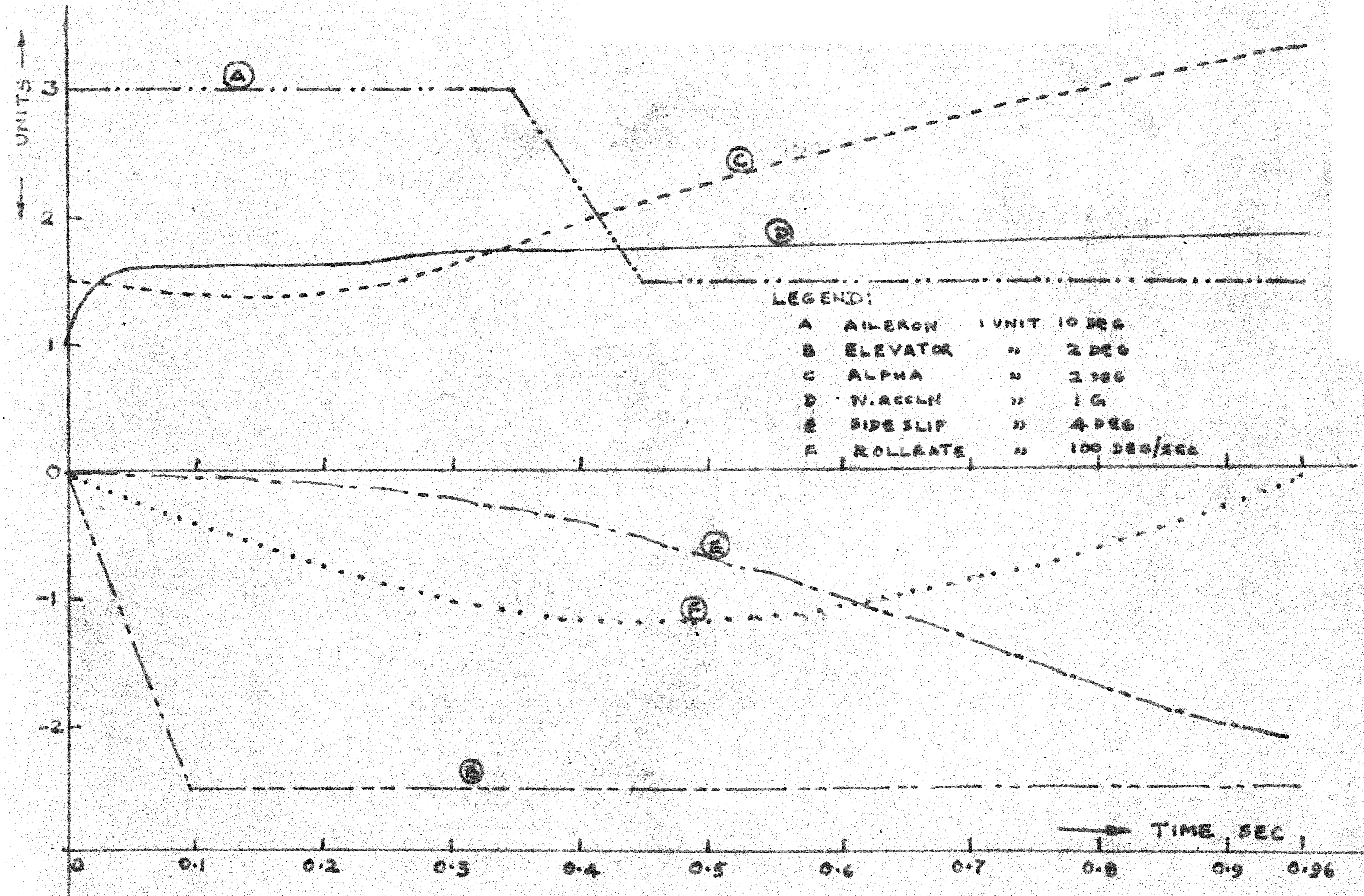

FIG 5.4 OPTIMAL CONTROLS AND TRAJECTORIES _ FLIGHT CONDITION C

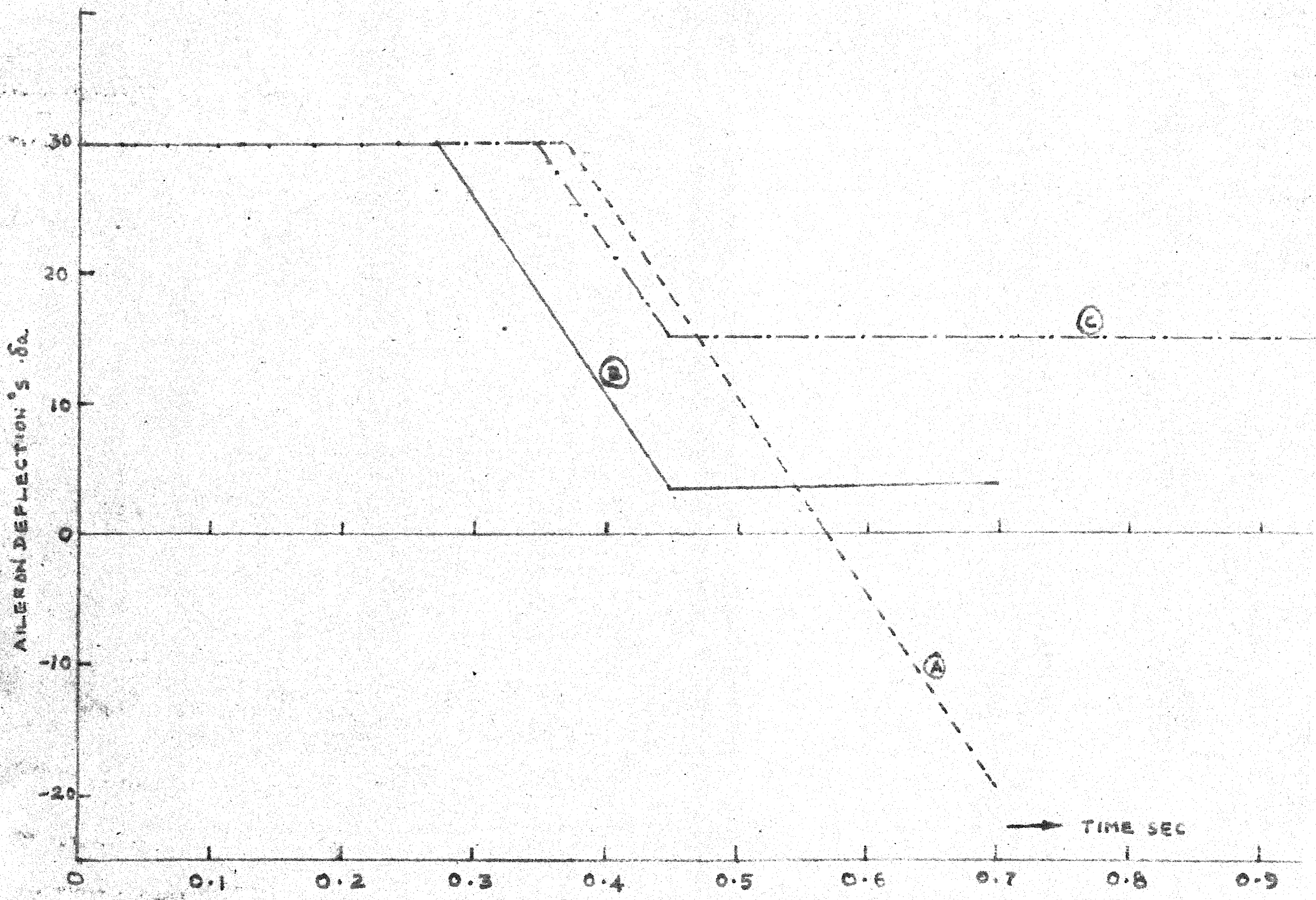

FIG 5.5 OPTIMAL AILERON CONTROLS FOR THE THREE FLIGHT CONDITIONS

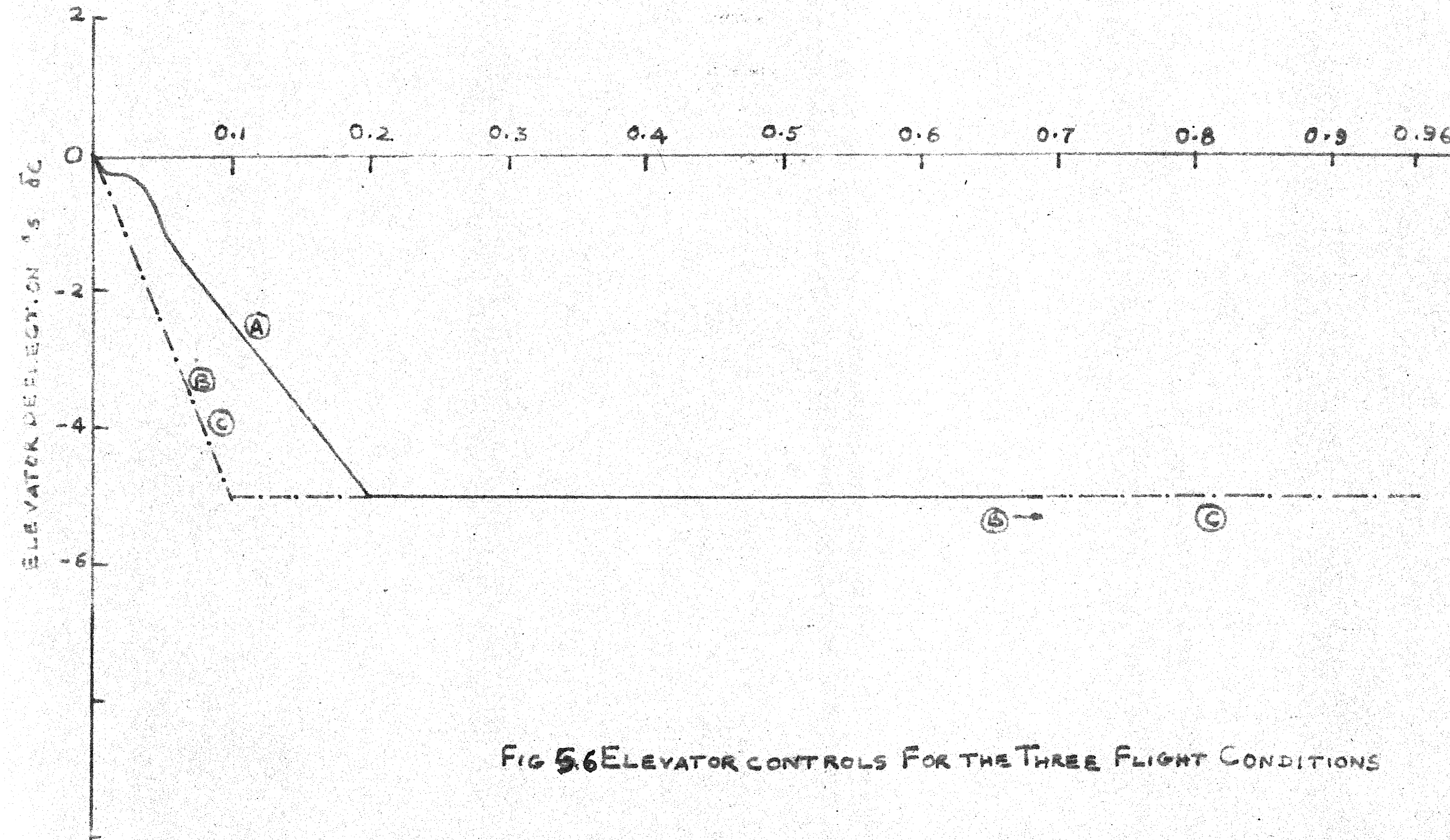

FIG 5.6 ELEVATOR CONTROLS FOR THE THREE FLIGHT CONDITIONS

CHAPTER VI

CONCLUSIONS

The problem of designing an optimal controller for performing the rolling pullout maneuver in a modern fighter aircraft was formulated. The effects of inertial cross coupling were brought out and the need for a nonlinear model including inertial cross coupling terms was emphasized. With this model and a suitable performance index, the maneuver problem was converted to an optimal control problem. A numerical method of solving this problem was presented. An example problem was solved for a modern fighter aircraft for three different flight conditions. The implementation aspects of the optimal control law and extension to various initial conditions were also given.

Scope for further research

Due to nonavailability of sufficient data, it was not possible to solve the problem for more than three flight conditions. With the results obtained for these three flight conditions, the nature of the control law is obvious; but with these results alone it is not possible to define the switching points of the control as a function of the Mach number and altitude. If sufficient data for evaluating the stability derivatives for all flight

conditions were available, the following topics merit further study :

(1) To find optimal control for various flight conditions covering the entire flight regime of the aircraft and try to arrive at an empirical relation for the control in terms of air speed and altitude.

(2) To study the effects of various initial conditions on the results obtained in (1).

(3) To design suitable logical circuits to take care of altitude, air speed and initial conditions.

APPENDIX A

CONVENTIONS AND DEFINITIONS

The sign conventions and definitions of certain terms used in this thesis are given below.

NACA STABILITY AXES : To specify the position of a moving aircraft, it is necessary to adopt an axes system. The axes system adopted in this thesis is according to NACA convention and is known as the "Stability Axes System". These are three mutually perpendicular axes with their origin coinciding with the centre of gravity of the aircraft. The axes system is fixed to the aircraft and moves with the aircraft. OX is positive forward and lies in the plane of symmetry of the aircraft with OZ also in the same plane with positive downward. OY is perpendicular to this plane of symmetry and is positive through the right wing. The axes system is shown clearly in Figure A-1. For a given initial flight condition, OX is fixed parallel to the relative wind vector at the start of the maneuver.

SPACE AXES : This is a set of mutually perpendicular axes (OX_0, OY_0, OZ_0) with OX_0 and OY_0 axes in the horizontal plane and the origin fixed at a particular point in the space.

DISPLACEMENT ANGLES :

(1) Pitch angle θ : It is the angle made by the X stability axis with its projection on the horizontal plane.

(2) Roll angle ϕ : It is the angle through which the stability axes system must be rotated about the X stability axis to make the Y stability axis parallel to horizontal.

(3) Yaw angle ψ : It is the angle between the projection of X stability axis on the horizontal plane and the X_o space axis.

SOME SIGN CONVENTIONS :

(1) A linear displacement along a positive reference axis is considered to be positive.

(2) An angular displacement or movement is considered positive if clockwise when viewed from the origin along a positive reference axis.

(3) Both linear and angular acceleration and velocities are considered positive in the same sense as their corresponding displacements.

(4) Control surface deflections are considered in the same sense as angular displacements. Ailerons move in opposite directions; hence the motion of the right aileron has been chosen to determine the sign of

the aileron angle. Therefore, right aileron down and left aileron up are considered positive and cause a negative roll according to the sign conventions for angular displacements.

RELATION BETWEEN MOVING AND FIXED AXES SYSTEMS

Since the stability axes move in space, to enable proper definition of angular velocities and displacements of the aircraft, the space axes are to be rotated in a particular sequence to orient them with the stability axes. The sequence is not particularly important as long as the same sequence is used through out. In this thesis, yaw pitch roll sequence is used for orientation of the space axes with the instantaneous position of stability axes. The orientation is done in the following manner :

(1) The "space axes" (OX_o, OY_o, OZ_o) are defined at the origin of the "stability axes system" as shown in Figure A-1.

(2) The space axes system is rotated about the Z_o axis through an angle ψ (yaw) to the orientation $X_2Y_2Z_2$. (The X_oY_o plane is rotated through an angle ψ.). The angular velocity about Z_o is $\dot{\psi}$.

(3) From the new orientation $X_2Y_2Z_2$, the space axes system is rotated about the Y_2 axis (pitch)

through an angle θ to the new position $X_3Y_3Z_3$. The angular velocity about Y_2 axis is $\dot{\theta}$.

(4) From the position assumed in (3) above, the axes system is rotated about X_3 axis (roll) to the final orientation XYZ. The displacement angle is ϕ and the angular velocity about the X axis is $\dot{\phi}$.

$\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$ IN TERMS OF p, q, r :

Let p, q, r be the total angular velocities of the aircraft (thus of the stability axes system) about the X, Y and Z stability axes respectively. Resolving $\dot{\phi}$, $\dot{\theta}$ and along the stability axes, we get

$$\dot{\phi} = p + q \sin\phi \tan\theta + r \cos\phi \tan\theta \qquad (A.1)$$

$$\dot{\theta} = q \cos\phi - r \sin\phi \qquad (A.2)$$

$$\dot{\psi} = (q \sin\phi + r \cos\phi)/\cos\theta$$

ϕ, θ and ψ can be obtained by integrating the above equations.

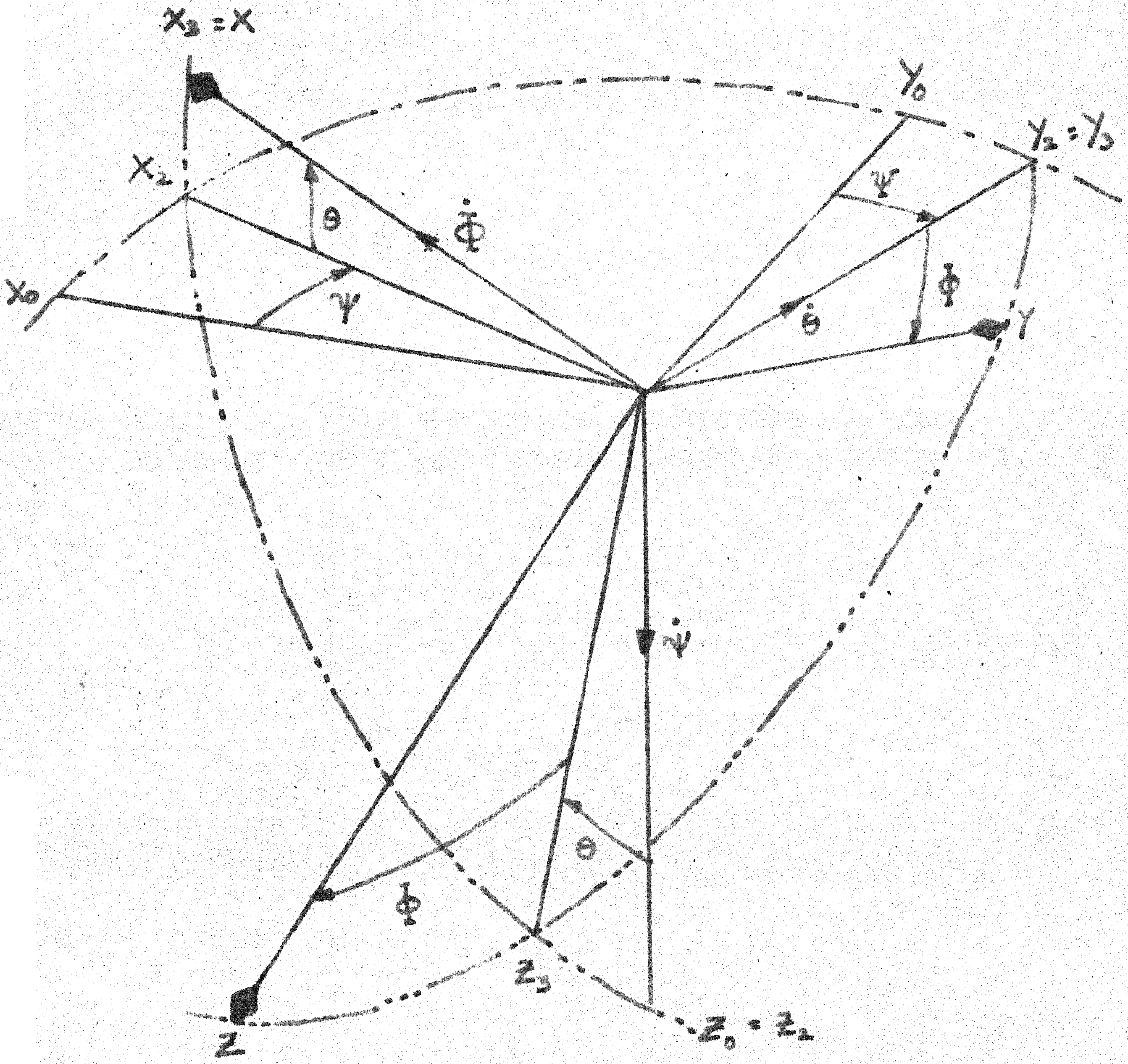

FIG A1. RELATION BETWEEN MOVING AND FIXED AXES

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